A class of II_1 factors with exactly two group measure space decompositions

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Abstract

We construct the first II_1 factors having exactly two group measure space decompositions up to unitary conjugacy. Also, for every positive integer n, we construct a II_1 factor M that has exactly n group measure space decompositions up to conjugacy by an automorphism.

1 Introduction

The *group measure space construction* of Murray and von Neumann associates to every free ergodic probability measure preserving (pmp) group action $\Gamma \curvearrowright (X, \mu)$ a crossed product von Neumann algebra $L^\infty(X) \rtimes \Gamma$. The classification of these group measure space II_1 factors is one of the core problems in operator algebras. For Γ infinite amenable, they are all isomorphic to the hyperfinite II_1 factor R, because by Connes' celebrated theorem [Co75], even all amenable II_1 factors are isomorphic with R.

For nonamenable groups Γ, rigidity phenomena appear and far reaching classification theorems for group measure space II_1 factors were established in Popa's deformation/rigidity theory, see e.g. [Po01, Po03, Po04]. In these results, the subalgebra $A = L^\infty(X)$ of $M = A \rtimes \Gamma$ plays a special role. Indeed, if an isomorphism $\pi : A \rtimes \Gamma \to B \rtimes \Lambda$ of group measure space II_1 factors satisfies $\pi(A) = B$, by [Si55], π must come from an orbit equivalence between the group actions $\Gamma \curvearrowright A$ and $\Lambda \curvearrowright B$, so that methods from measured group theory can be used. The subalgebra $A \subset M$ is *Cartan:* it is maximal abelian and the normalizer $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^*\}$ generates M. One of the key results of [Po01, Po03, Po04] is that for large classes of group actions, any isomorphism $\pi : A \rtimes \Gamma \to B \rtimes \Lambda$ satisfies $\pi(A) = B$, up to unitary conjugacy.

In [Pe09, PV09], the most extreme form of rigidity, called *W^*-superrigidity* was discovered: in certain cases, the crossed product II_1 factor $M = A \rtimes \Gamma$ entirely remembers Γ and its action on A. These results were established by first proving that the II_1 factor M has a *unique group measure space Cartan subalgebra* up to unitary conjugacy and then proving that the group action $\Gamma \curvearrowright A$ is OE-superrigid. Since then, several uniqueness results for group measure space Cartan subalgebras were obtained, in particular [Io10] proving that all Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ of all infinite property (T) groups are W^*-superrigid.

Note that a general Cartan subalgebra $A \subset M$ need not be of group measure space type, i.e. there need not exist a group Γ complementing A in such a way that $M = A \rtimes \Gamma$. This is closely related to the phenomenon that a countable pmp equivalence relation need not be the orbit equivalence relation of a group action that is free. The first actual uniqueness theorems for Cartan subalgebras up to unitary conjugacy were only obtained in [OP07], where it was proved in particular that A is the unique Cartan subalgebra of $A \rtimes \Gamma$ whenever $\Gamma = \mathbb{F}_n$ is a free group and $\mathbb{F}_n \curvearrowright A$ is a free ergodic pmp action that is *profinite*. More recently, in [PV11], it was

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shown that A is the unique Cartan subalgebra of $A \rtimes \Gamma$ for *arbitrary* free ergodic pmp actions of the free groups $\Gamma = \mathbb{F}_n$.

It is known since [CJ82] that a II_1 factor M may have two Cartan subalgebras $A, B \subset M$ that are non conjugate by an automorphism of M. Although several concrete examples of this phenomenon were given in [OP08, PV09, SV11] and despite all the progress on uniqueness of Cartan subalgebras, there are so far no results describing *all* Cartan subalgebras of a II_1 factor M once uniqueness fails. In this paper, we prove such a result and the following is our main theorem.

Theorem A. (1) *For every integer $n \geq 0$, there exist II_1 factors M that have exactly 2^n group measure space Cartan subalgebras up to unitary conjugacy.*

(2) *For every integer $n \geq 1$, there exist II_1 factors M that have exactly n group measure space Cartan subalgebras up to conjugacy by an automorphism of M.*

Two free ergodic pmp actions are called *W^*-equivalent* if they have isomorphic crossed product von Neumann algebras. Thus, a free ergodic pmp action $G \curvearrowright (X, \mu)$ is W*-superrigid if every action that is W*-equivalent to $G \curvearrowright (X, \mu)$ must be conjugate to $G \curvearrowright (X, \mu)$. Theorem A(2) can then be rephrased in the following way: we construct free ergodic pmp actions $G \curvearrowright (X, \mu)$ with the property that $G \curvearrowright (X, \mu)$ is W*-equivalent to exactly n group actions, up to orbit equivalence of the actions (and actually also up to conjugacy of the actions, see Theorem 6.3).

The II_1 factors M in Theorem A are concretely constructed as follows. Let Γ be any torsion free nonelementary hyperbolic group and let $\beta : \Gamma \curvearrowright (A_0, \tau_0)$ be any trace preserving action on the amenable von Neumann algebra (A_0, τ_0) with $A_0 \neq \mathbb{C}1$ and $\operatorname{Ker} \beta \neq \{e\}$. We then define $(A, \tau) = (A_0, \tau_0)^\Gamma$ and consider the action $\sigma : \Gamma \times \Gamma \curvearrowright (A, \tau)$ given by $\sigma_{(g,h)}(\pi_k(a)) = \pi_{gkh^{-1}}(\beta_h(a))$ for all $g, h, k \in \Gamma$ and $a \in A_0$, where $\pi_k : A_0 \to A$ denotes the embedding as the k'th tensor factor.

Our main result describes exactly all group measure space Cartan subalgebras of the crossed product $M = A_0^\Gamma \rtimes (\Gamma \times \Gamma)$.

Theorem B. *Let $M = A_0^\Gamma \rtimes (\Gamma \times \Gamma)$ be as above. Up to unitary conjugacy, all group measure space Cartan subalgebras $B \subset M$ are of the form $B = B_0^\Gamma$ where $B_0 \subset A_0$ is a group measure space Cartan subalgebra of A_0 with the following two properties: $\beta_g(B_0) = B_0$ for all $g \in \Gamma$ and A_0 can be decomposed as $A_0 = B_0 \rtimes \Lambda_0$ with $\beta_g(\Lambda_0) = \Lambda_0$ for all $g \in \Gamma$.*

In Section 5, we actually prove a more general and more precise result, see Theorem 5.1. In Section 6, we give concrete examples and computations, thus proving Theorem A (see Theorem 6.3).

Note that in Theorems A and B, we can only describe the group measure space Cartan subalgebras of M. The reason for this is that our method entirely relies on a technique of [PV09], using the so-called *dual coaction* that is associated to a group measure space decomposition $M = B \rtimes \Lambda$, i.e. the normal $*$-homomorphism $\Delta : M \to M \overline{\otimes} M$ given by $\Delta(bv_s) = bv_s \otimes v_s$ for all $b \in B$, $s \in \Lambda$. When $B \subset M$ is an arbitrary Cartan subalgebra, we do not have such a structural $*$-homomorphism.

Given a II_1 factor M as in Theorem B and given the dual coaction $\Delta : M \to M \overline{\otimes} M$ associated with an arbitrary group measure space decomposition $M = B \rtimes \Lambda$, Popa's key methods of malleability [Po03] and spectral gap rigidity [Po06] for Bernoulli actions allow to prove that $\Delta(L(\Gamma \times \Gamma))$ can be unitarily conjugated into $M \overline{\otimes} L(\Gamma \times \Gamma)$. An ultrapower technique of [Io11], in combination with the transfer-of-rigidity principle of [PV09], then shows that the "mysterious" group Λ must contain two commuting nonamenable subgroups Λ_1, Λ_2. Note here that the same combination of [Io11] and [PV09] was recently used in [CdSS15] to prove that if Γ_1, Γ_2

are nonelementary hyperbolic groups and $L(\Gamma_1 \times \Gamma_2) \cong L(\Lambda)$, then Λ must be a direct product of two nonamenable groups. Once we know that Λ contains two commuting nonamenable subgroups Λ_1, Λ_2, we use a combination of methods from [Io10] and [IPV10] to prove that $\Lambda_1\Lambda_2 \subset \Gamma \times \Gamma$. From that point on, it is not so hard any more to entirely unravel the structure of B and Λ. Throughout these arguments, we repeatedly use the crucial dichotomy theorem of [PV11, PV12] saying that hyperbolic groups Γ are relatively strongly solid: in arbitrary tracial crossed products $M = P \rtimes \Gamma$, if a von Neumann subalgebra $Q \subset M$ is amenable relative to P, then either Q embeds into P, or the normalizer of Q stays amenable relative to P.

2 Preliminaries

2.1 Popa's intertwining-by-bimodules

We recall from [Po03, Theorem 2.1 and Corollary 2.3] Popa's method of intertwining-by-bimodules. When (M, τ) is a tracial von Neumann algebra and $P, Q \subset M$ are possibly non-unital von Neumann subalgebras, we write $P \prec_M Q$ if there exists a nonzero P-Q-subbimodule of $1_P L^2(M) 1_Q$ that has finite right Q-dimension. We refer to [Po03] for several equivalent formulations of this intertwining property. If $Pp \prec_M Q$ for all nonzero projections $p \in P' \cap 1_P M 1_P$, we write $P \prec^f_M Q$.

We are particularly interested in the case where M is a crossed product $M = A \rtimes \Gamma$ by a trace preserving action $\Gamma \curvearrowright (A, \tau)$. Given a subset $F \subset \Gamma$, we denote by P_F the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{a u_g \mid a \in A, g \in F\}$, where $\{u_g\}_{g\in\Gamma}$ denote the canonical unitaries in $L(\Gamma)$. By [Va10, Lemma 2.5], a von Neumann subalgebra $P \subset pMp$ satisfies $P \prec^f_M A$ if and only if for every $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma$ such that

$$\|x - P_F(x)\|_2 \leq \|x\|\varepsilon \quad \text{for all } x \in P \; .$$

We also need the following elementary lemma.

Lemma 2.1. *Let $\Gamma \curvearrowright (A, \tau)$ be a trace preserving action and put $M = A \rtimes \Gamma$. If $P \subset M$ is a diffuse von Neumann subalgebra such that $P \prec^f A$, then $P \not\prec L(\Gamma)$.*

Proof. Let $\varepsilon > 0$ be given. We have $P \prec^f A$. So, as explained above, we can take a finite set $F \subset \Gamma$ such that $\|u - P_F(u)\|_2 \leq \frac{\varepsilon}{2}$ for all $u \in \mathcal{U}(P)$. Moreover, since P is diffuse, we can choose a sequence of unitaries $(w_n) \subset \mathcal{U}(P)$ tending to 0 weakly. We will prove that $\|E_{L(\Gamma)}(x w_n y)\|_2 \to 0$ for all $x, y \in M$, meaning that $P \not\prec L(\Gamma)$. Note that it suffices to consider $x, y \in (A)_1$.

Take $x, y \in (A)_1$. Write $w_n = \sum_{g\in\Gamma} (w_n)_g u_g$ with $(w_n)_g \in A$. Then,

$$\|E_{L(\Gamma)}(P_F(x w_n y))\|_2^2 = \sum_{g\in F} |\tau(x (w_n)_g \sigma_g(y))|^2 \to 0$$

since $w_n \to 0$ weakly. Take n_0 large enough such that

$$\|E_{L(\Gamma)}(P_F(x w_n y))\|_2 \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq n_0 \; .$$

Since P_F is A-A-bimodular, we have that

$$\|E_{L(\Gamma)}(P_F(x w_n y)) - E_{L(\Gamma)}(x w_n y)\|_2 \leq \|P_F(x w_n y) - x w_n y\|_2 = \|x (P_F(w_n) - w_n) y\|_2 \leq \frac{\varepsilon}{2}$$

for all n. We conclude that $\|E_{L(\Gamma)}(x w_n y)\|_2 \leq \varepsilon$ for all $n \geq n_0$. ☐

2.2 Relative amenability

A tracial von Neumann algebra (M, τ) is called *amenable* if there exists a state φ on $B(L^2(M))$ such that $\varphi|_M = \tau$ and such that φ is M-central, meaning that $\varphi(xT) = \varphi(Tx)$ for all $x \in M$, $T \in B(L^2(M))$. In [OP07, Section 2.2], the concept of relative amenability was introduced. The definition makes use of Jones' basic construction: given a tracial von Neumann algebra (M, τ) and a von Neumann subalgebra $P \subset M$, the basic construction $\langle M, e_P \rangle$ is defined as the commutant of the right P-action on $B(L^2(M))$. Following [OP07, Definition 2.2], we say that a von Neumann subalgebra $Q \subset pMp$ is *amenable relative to P inside M* if there exists a positive functional φ on $p\langle M, e_P \rangle p$ that is Q-central and satisfies $\varphi|_{pMp} = \tau$.

We say that Q is *strongly nonamenable relative to P* if Qq is nonamenable relative to P for every nonzero projection $q \in Q' \cap pMp$. Note that in that case, also qQq is strongly nonamenable relative to P for all nonzero projections $q \in Q$.

We need the following elementary relationship between relative amenability and intertwining-by-bimodules.

Proposition 2.2. *Let (M, τ) be a tracial von Neumann algebra and $Q, P_1, P_2 \subset M$ be von Neumann subalgebras with $P_1 \subset P_2$. Assume that Q is strongly nonamenable relative to P_1. Then the following holds.*

(1) *If $Q \prec P_2$, there exist projections $q \in Q$, $p \in P_2$, a nonzero partial isometry $v \in qMp$ and a normal unital $*$-homomorphism $\theta : qQq \to pP_2p$ such that $xv = v\theta(x)$ for all $x \in qQq$ and such that, inside P_2, we have that $\theta(qQq)$ is nonamenable relative to P_1.*

(2) *We have $Q \not\prec P_1$.*

Proof. (1) Assume that $Q \prec P_2$. By [Po03, Theorem 2.1], we can take projections $q \in Q$, $p \in P_2$, a nonzero partial isometry $v \in qMp$ and a normal unital $*$-homomorphism $\theta : qQq \to pP_2p$ such that $xv = v\theta(x)$ for all $x \in qQq$. Assume that $\theta(qQq)$ is amenable relative to P_1 inside P_2. We can then take a positive functional φ on $p\langle P_2, e_{P_1} \rangle p$ that is $\theta(qQq)$-central and satisfies $\varphi|_{pP_2p} = \tau$. Denote by e_{P_2} the orthogonal projection of $L^2(M)$ onto $L^2(P_2)$. Observe that $e_{P_2}\langle M, e_{P_1} \rangle e_{P_2} = \langle P_2, e_{P_1} \rangle$. We can then define the positive functional ω on $q\langle M, e_{P_1} \rangle q$ given by

$$\omega(T) = \varphi(e_{P_2} v^* T v e_{P_2}) \quad \text{for all } \; T \in q\langle M, e_{P_1} \rangle q \; .$$

By construction, ω is qQq-central and $\omega(x) = \tau(v^* x v)$ for all $x \in qMq$. Writing $q_0 = vv^*$, we have $q_0 \in (Q' \cap M)q$ and it follows that $qQqq_0$ is amenable relative to P_1. This contradicts the strong nonamenability of Q relative to P_1.

Finally, note that (2) follows from (1) by taking $P_1 = P_2$. $\square$

2.3 Properties of the dual coaction

Let $M = B \rtimes \Lambda$ be any tracial crossed product von Neumann algebra. Denote by $\{v_s\}_{s \in \Lambda}$ the canonical unitaries and consider the *dual coaction* $\Delta : M \to M \mathbin{\overline{\otimes}} M$, i.e. the normal $*$-homomorphism defined by $\Delta(b) = b \otimes 1$ for all $b \in B$ and $\Delta(v_s) = v_s \otimes v_s$ for all $s \in \Lambda$.

First, we show the following elementary lemma.

Lemma 2.3. *A von Neumann subalgebra $A \subset B \rtimes \Lambda$ satisfies $\Delta(A) \subset A \mathbin{\overline{\otimes}} A$ if and only if $A = B_0 \rtimes \Lambda_0$ for some von Neumann subalgebra $B_0 \subset B$ and some subgroup $\Lambda_0 < \Lambda$ that leaves B_0 globally invariant.*

Proof. Let $A \subset B \rtimes \Lambda$ be a von Neumann subalgebra satisfying $\Delta(A) \subset A \mathbin{\overline{\otimes}} A$. Let $a \in A$ and write $a = \sum_{s \in \Lambda} a_s v_s$ with $a_s \in B$. Let $I = \{s \in \Lambda \mid a_s \neq 0\}$. Fix $s \in I$ and define the normal linear functional ω on $B \rtimes \Lambda$ by $\omega(x) = \tau(x v_s^* a_s^*)$. Then $(\omega \otimes 1)\Delta(a) = \|a_s\|_2^2 \, v_s$. Since $\Delta(a) \in A \mathbin{\overline{\otimes}} A$, it follows that $v_s \in A$. Similarly, we define a linear functional ρ on $B \rtimes \Lambda$ by $\rho(x) = \tau(x v_s^*)$. Then $(1 \otimes \rho)\Delta(a) = a_s v_s \in A$ and it follows that $a_s \in A$. Since this holds for arbitrary $s \in I$, we conclude that $A = B_0 \rtimes \Lambda_0$ where $B_0 = A \cap B$ and $\Lambda_0 = \{s \in \Lambda \mid v_s \in A\}$. $\square$

The proof of the next result is almost identical to the proof of [IPV10, Lemma 7.2(4)]. For the convenience of the reader, we provide some details.

Proposition 2.4. *Assume that (B, τ) is amenable. If $Q \subset M$ is a von Neumann subalgebra without amenable direct summand, then $\Delta(Q)$ is strongly nonamenable relative to $M \otimes 1$.*

Proof. Using the bimodule characterization of relative amenability (see [OP07, Theorem 2.1] and see also [PV11, Proposition 2.4]), it suffices to prove that the $(M \mathbin{\overline{\otimes}} M)$-$M$-bimodule

$$_{M \overline{\otimes} M}\big(L^2(M \mathbin{\overline{\otimes}} M) \underset{M \otimes 1}{\otimes} L^2(M \mathbin{\overline{\otimes}} M)\big)_{\Delta(M)}$$

is weakly contained in the coarse $(M \mathbin{\overline{\otimes}} M)$-$M$-bimodule. Denoting by $\sigma : M \mathbin{\overline{\otimes}} M \to M \mathbin{\overline{\otimes}} M$ the flip automorphism, this bimodule is isomorphic with the $(M \mathbin{\overline{\otimes}} M)$-$M$-bimodule

$$_{M \overline{\otimes} M} L^2(M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M)_{(\mathrm{id} \otimes \sigma)(\Delta(M) \otimes 1)} \; .$$

So, it suffices to prove that the M-M-bimodule $_{M \otimes 1} L^2(M \mathbin{\overline{\otimes}} M)_{\Delta(M)}$ is weakly contained in the coarse M-M-bimodule. Noting that this M-M-bimodule is isomorphic with a multiple of the M-M-bimodule $_M\big(L^2(M) \otimes_B L^2(M)\big)_M$, the result follows from the amenability of B. $\square$

2.4 Spectral gap rigidity for co-induced actions

Given a tracial von Neumann algebra (A_0, τ_0) and a countable set I, we denote by $(A_0, \tau_0)^I$ (or just A_0^I) the von Neumann algebra tensor product $\overline{\bigotimes}_I (A_0, \tau_0)$. For each $i \in I$, we denote by $\pi_i \colon A_0 \to A_0^I$ the embedding of A_0 as the i'th tensor factor.

Definition 2.5. Let $\Gamma \curvearrowright I$ be an action of a countable group Γ on a countable set I. We say that a trace preserving action $\sigma \colon \Gamma \curvearrowright (A_0, \tau_0)^I$ is *built over* $\Gamma \curvearrowright I$ if it satisfies

$$\sigma_g(\pi_i(A_0)) = \pi_{g \cdot i}(A_0) \quad \text{for all } \; g \in \Gamma, \, i \in I \; .$$

Assume that $\Gamma \curvearrowright A_0^I$ is an action built over $\Gamma \curvearrowright I$. Choose a subset $J \subset I$ that contains exactly one point in every orbit of $\Gamma \curvearrowright I$. For every $j \in J$, the group $\operatorname{Stab} j$ globally preserves $\pi_j(A_0)$. This defines an action $\operatorname{Stab} j \curvearrowright A_0$ that can be *co-induced* to an action $\Gamma \curvearrowright A_0^{\Gamma / \operatorname{Stab} i}$. The original action $\Gamma \curvearrowright A_0^I$ is conjugate with the direct product of all these co-induced actions. In particular, co-induced actions are exactly actions built over a transitive action $\Gamma \curvearrowright I = \Gamma / \Gamma_0$.

Popa's malleability [Po03] and spectral gap rigidity [Po06] apply to actions built over $\Gamma \curvearrowright I$. The generalization provided in [BV14, Theorems 3.1 and 3.3] carries over verbatim and this gives the following result.

Theorem 2.6. *Let $\Gamma \curvearrowright I$ be an action of an icc group on a countable set. Assume that* $\operatorname{Stab}\{i, j\}$ *is amenable for all $i, j \in I$ with $i \neq j$. Let (A_0, τ_0) be a tracial von Neumann algebra and (N, τ) a II$_1$ factor. Let $\Gamma \curvearrowright (A_0, \tau_0)^I$ be an action built over $\Gamma \curvearrowright I$ and put $M = A_0^I \rtimes \Gamma$. If $P \subset N \mathbin{\overline{\otimes}} M$ is a von Neumann subalgebra that is strongly nonamenable relative to $N \mathbin{\overline{\otimes}} A_0^I$, then the relative commutant $Q := P' \cap N \mathbin{\overline{\otimes}} M$ satisfies at least one of the following properties:*

(1) *there exists an* $i \in I$ *such that* $Q \prec N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab} i)$ *;*
(2) *there exists a unitary* $v \in N \mathbin{\overline{\otimes}} M$ *such that* $v^* Q v \subset N \mathbin{\overline{\otimes}} L(\Gamma)$.

Recall that for a von Neumann subalgebra $P \subset M$, we define $\mathcal{QN}_M(P) \subset M$ as the set of elements $x \in M$ for which there exist $x_1, \ldots, x_n, y_1, \ldots, y_m \in M$ satisfying

$$xP \subset \sum_{i=1}^{n} P x_i \quad \text{and} \quad Px \subset \sum_{j=1}^{m} y_j P \; .$$

Then $\mathcal{QN}_M(P)$ is a $*$-subalgebra of M containing P. Its weak closure is called the *quasi-normalizer* of P inside M.

The following lemma is proved in exactly the same way as [IPV10, Lemma 4.1] and goes back to [Po03, Theorem 3.1].

Lemma 2.7. *Let* $\Gamma \curvearrowright I$ *be an action. Let* A_0 *be a tracial von Neumann algebra and let* $\Gamma \curvearrowright A_0^I$ *be an action built over* $\Gamma \curvearrowright I$. *Consider the crossed product* $M = A_0^I \rtimes \Gamma$ *and let* (N, τ) *be an arbitrary tracial von Neumann algebra.*

(1) *If* $P \subset p(N \mathbin{\overline{\otimes}} L(\Gamma))p$ *is a von Neumann subalgebra such that* $P \not\prec_{N \overline{\otimes} L(\Gamma)} N \mathbin{\overline{\otimes}} L(\operatorname{Stab} i)$ *for all* $i \in I$, *then the quasi-normalizer of* P *inside* $p(N \mathbin{\overline{\otimes}} M)p$ *is contained in* $p(N \mathbin{\overline{\otimes}} L(\Gamma))p$.
(2) *Fix* $i \in I$ *and assume that* $Q \subset q(N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab} i))q$ *is a von Neumann subalgebra such that for all* $j \neq i$, *we have* $Q \not\prec_{N \overline{\otimes} (A_0^I \rtimes \operatorname{Stab} i)} N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab}\{i, j\})$. *Then the quasi-normalizer of* Q *inside* $q(N \mathbin{\overline{\otimes}} M)q$ *is contained in* $q(N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab} i))q$.

3 Transfer of rigidity

Fix a trace preserving action $\Lambda \curvearrowright (B, \tau)$ and put $M = B \rtimes \Lambda$. We denote by $\{v_s\}_{s \in \Lambda}$ the canonical unitaries in $L(\Lambda)$. Whenever $\mathcal{G}$ is a family of subgroups of Λ, we say that a subset $F \subset \Lambda$ is small relative to $\mathcal{G}$ if F is contained in a finite union of subsets of the form $g \Sigma h$ where $g, h \in \Lambda$ and $\Sigma \in \mathcal{G}$.

Following the transfer of rigidity principle from [PV09, Section 3], we prove the following theorem.

Theorem 3.1. *Let* $\Lambda \curvearrowright (B, \tau)$ *be a trace preserving action and put* $M = B \rtimes \Lambda$. *Let* $\Delta \colon M \to M \mathbin{\overline{\otimes}} M$ *be the dual coaction given by* $\Delta(b v_s) = b v_s \otimes v_s$ *for* $b \in B, s \in \Lambda$. *Let* $\mathcal{G}$ *be a family of subgroups of* Λ. *Let* $P, Q \subset M$ *be two von Neumann subalgebras satisfying*

(1) $\Delta(P) \prec_{M \overline{\otimes} M} M \mathbin{\overline{\otimes}} Q$,
(2) $P \not\prec_M B \rtimes \Sigma$ *for all* $\Sigma \in \mathcal{G}$.

Then there exists a finite set $x_1, \ldots, x_n \in M$ *and a* $\delta > 0$ *such that the following holds: whenever* $F \subset \Lambda$ *is small relative to* $\mathcal{G}$, *there exists an element* $s_F \in \Lambda - F$ *such that*

$$\sum_{i,k=1}^{n} \| E_Q \left(x_i v_{s_F} x_k^* \right) \|_2^2 \geq \delta \; .$$

Proof. Since $\Delta(P) \prec_{M \overline{\otimes} M} M \mathbin{\overline{\otimes}} Q$, we can find a finite set $x_1, \ldots, x_n \in M$ and $\rho > 0$ such that

$$\sum_{i,k=1}^{n} \| E_{M \overline{\otimes} Q} \left((1 \otimes x_i) \Delta(w) (1 \otimes x_k^*) \right) \|_2^2 \geq \rho \quad \text{for all } w \in \mathcal{U}(P) \; .$$

Given a subset $F \subset \Lambda$, we denote by P_F the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{bv_s \mid b \in B, s \in F\}$. Since $P \not\prec_M B \rtimes \Sigma$ for all $\Sigma \in \mathcal{G}$, it follows from [Va10, Lemma 2.4] that there exists a net of unitaries $\{w_j\}_{j \in J} \subset \mathcal{U}(P)$ such that $\|P_F(w_j)\|_2 \to 0$ for any set $F \subset \Lambda$ that is small relative to $\mathcal{G}$. For each $j \in J$, write $w_j = \sum_{s \in \Lambda} w_s^j v_s$ with $w_s^j \in B$ and compute

$$\rho \leq \sum_{i,k=1}^{n} \|E_{M \overline{\otimes} Q}\left((1 \otimes x_i)\Delta(w_j)(1 \otimes x_k^*)\right)\|_2^2 = \sum_{i,k=1}^{n} \left\| \sum_{s \in \Lambda} w_s^j v_s \otimes E_Q(x_i v_s x_k^*) \right\|_2^2$$
$$= \sum_{i,k=1}^{n} \sum_{s \in \Lambda} \|w_s^j\|_2^2 \, \|E_Q(x_i v_s x_k^*)\|_2^2 \, .$$

We now claim that the conclusion of the theorem holds with $\delta = \frac{\rho}{2}$. Indeed, assume for contradiction that there exists a subset $F \subset \Lambda$ that is small relative to $\mathcal{G}$ such that

$$\sum_{i,k=1}^{n} \|E_Q\left(x_i v_s x_k^*\right)\|_2^2 < \delta \quad \text{for all } s \in \Lambda - F \, .$$

Put $K = \max\{\|x_i\|^2 \|x_k^*\|_2^2 \mid i,k = 1,\ldots,n\}$ and choose $j_0 \in J$ such that $\|P_F(w_j)\|_2^2 = \sum_{s \in F} \|w_s^j\|_2^2 < \frac{\rho}{4Kn^2}$ for all $j \geq j_0$. We then get for $j \geq j_0$

$$\rho \leq \sum_{s \in \Lambda} \sum_{i,k=1}^{n} \|w_s^j\|_2^2 \, \|E_Q(x_i v_s x_k^*)\|_2^2 \leq n^2 K \sum_{s \in F} \|w_s^j\|_2^2 + \sum_{s \in \Lambda - F} \|w_s^j\|_2^2 \, \delta < \frac{\rho}{4} + \frac{\rho}{2} \, ,$$

which is a contradiction. □

4 Embeddings of group von Neumann algebras

Following [Io10, Section 4] and [IPV10, Section 3], we define the *height* h_Γ of an element in a group von Neumann algebra $L(\Gamma)$ as the absolute value of the largest Fourier coefficient, i.e.,

$$h_\Gamma(x) = \sup_{g \in \Gamma} |\tau(x u_g^*)| \quad \text{for } x \in L(\Gamma) \, .$$

Whenever $\mathcal{G} \subset L(\Gamma)$, we write

$$h_\Gamma(\mathcal{G}) = \inf\{h_\Gamma(x) \mid x \in \mathcal{G}\} \, .$$

When Γ is an icc group and Λ is a countable group such that $L(\Lambda) = L(\Gamma)$ with $h_\Gamma(\Lambda) > 0$, it was proven in [IPV10, Theorem 3.1] that there exists a unitary $u \in L(\Gamma)$ such that $u\mathbb{T}\Lambda u^* = \mathbb{T}\Gamma$. We need the following generalization. For this, recall that a unitary representation is said to be *weakly mixing* if $\{0\}$ is the only finite dimensional subrepresentation.

Theorem 4.1. *Let Γ be a countable group and $p \in L(\Gamma)$ a projection. Assume that $\mathcal{G} \subset \mathcal{U}(pL(\Gamma)p)$ is a subgroup satisfying the following properties.*

(1) *The unitary representation $\{\operatorname{Ad} v\}_{v \in \mathcal{G}}$ on $L^2(pL(\Gamma)p \ominus \mathbb{C}p)$ is weakly mixing.*

(2) *If $g \in \Gamma$ and $g \neq e$, then $\mathcal{G}'' \not\prec L(C_\Gamma(g))$.*

(3) *We have $h_\Gamma(\mathcal{G}) > 0$.*

Then $p = 1$ and there exists a unitary $u \in L(\Gamma)$ such that $u\mathcal{G}u^* \subset \mathbb{T}\Gamma$.

Proof. Write $M = L(\Gamma)$ and denote by $\Delta : M \to M \overline{\otimes} M : \Delta(u_g) = u_g \otimes u_g$ the comultiplication on $L(\Gamma)$. We first prove that the unitary representation on $L^2(\Delta(p)(M\overline{\otimes}M)\Delta(p) \ominus \Delta(\mathbb{C}p))$ given by $\{\operatorname{Ad}\Delta(v)\}_{v\in\mathcal{G}}$ is weakly mixing. To prove this, assume that $\mathcal{H} \subset L^2(\Delta(p)(M \overline{\otimes} M)\Delta(p))$ is a finite dimensional subspace satisfying $\Delta(v)\mathcal{H}\Delta(v^*) = \mathcal{H}$ for all $v \in \mathcal{G}$. Writing $P = \mathcal{G}''$, it follows that the closed linear span of $\mathcal{H}\Delta(pMp)$ is a $\Delta(P)$-$\Delta(pMp)$-subbimodule of $L^2(\Delta(p)(M \overline{\otimes} M)\Delta(p))$ that has finite right dimension. By [IPV10, Proposition 7.2] (using that $P \not\prec L(C_\Gamma(g))$ for $g \neq e$), we get that $\mathcal{H} \subset \Delta(L^2(pMp))$. Since the unitary representation $\{\operatorname{Ad} v\}_{v\in\mathcal{G}}$ on $L^2(pMp \ominus \mathbb{C}p)$ is weakly mixing, we conclude that $\mathcal{H} \subset \mathbb{C}\Delta(p)$.

Using the Fourier decomposition $v = \sum_{g\in\Gamma}(v)_g u_g$, we get for every $v \in \mathcal{G}$ that

$$\tau((v \otimes \Delta(v))(\Delta(v)^* \otimes v^*)) = \sum_{g\in\Gamma} |(v)_g|^4 \geq h_\Gamma(v)^4 \geq h_\Gamma(\mathcal{G})^4 \,.$$

Defining $X \in M \overline{\otimes} M \overline{\otimes} M$ as the element of minimal $\|\cdot\|_2$ in the weakly closed convex hull of $\{(v \otimes \Delta(v))(\Delta(v)^* \otimes v^*) \mid v \in \mathcal{G}\}$, we get that $\tau(X) \geq h_\Gamma(\mathcal{G})^4$, so that X is nonzero, and that $(v \otimes \Delta(v))X = X(\Delta(v) \otimes v)$ for all $v \in \mathcal{G}$. Also note that $(p \otimes \Delta(p))X = X = X(\Delta(p) \otimes p)$. By the weak mixing of both $\operatorname{Ad} v$ and $\operatorname{Ad}\Delta(v)$, it follows that XX^* is multiple of $p\otimes\Delta(p)$ and that X^*X is a multiple of $\Delta(p) \otimes p$. We may thus assume that

$$XX^* = p \otimes \Delta(p) \quad \text{and} \quad X^*X = \Delta(p) \otimes p \,.$$

Define $Y = (1 \otimes X)(X \otimes 1)$. Note that $Y \in M \overline{\otimes} M \overline{\otimes} M \overline{\otimes} M$ is a partial isometry with $YY^* = p \otimes p \otimes \Delta(p)$ and $Y^*Y = \Delta(p) \otimes p \otimes p$. Also,

$$Y = (v \otimes v \otimes \Delta(v))Y(\Delta(v)^* \otimes v^* \otimes v^*) \quad \text{for all} \;\; v \in \mathcal{G} \,.$$

Since Y is nonzero, it follows that the unitary representation $\xi \mapsto (v \otimes v)\xi\Delta(v^*)$ of $\mathcal{G}$ on the Hilbert space $(p \otimes p)L^2(M \overline{\otimes} M)\Delta(p)$ is not weakly mixing. We thus find a finite dimensional irreducible representation $\omega : \mathcal{G} \to \mathcal{U}(\mathbb{C}^n)$ and a nonzero $Z \in M_{n,1}(\mathbb{C}) \otimes (p\otimes p)L^2(M \overline{\otimes} M)\Delta(p)$ satisfying

$$(\omega(v) \otimes v \otimes v)Z = Z\Delta(v) \quad \text{for all} \;\; v \in \mathcal{G} \,.$$

By the weak mixing of $\operatorname{Ad} v$ and $\operatorname{Ad}\Delta(v)$ and the irreducibility of ω, it follows that ZZ^* is a multiple of $1\otimes p\otimes p$ and that Z^*Z is a multiple of $\Delta(p)$. So, we may assume that $ZZ^* = 1\otimes p\otimes p$ and that $Z^*Z = \Delta(p)$. It follows that $Z^*(M_n(\mathbb{C}) \otimes p \otimes p)Z$ is an n^2-dimensional globally $\{\operatorname{Ad}\Delta(v)\}_{v\in\mathcal{G}}$-invariant subspace of $\Delta(p)(M \overline{\otimes} M)\Delta(p)$. Again by weak mixing, this implies that $n = 1$. But then, since $\tau(ZZ^*) = \tau(Z^*Z)$, we also get that $p = 1$. So, $Z \in M \overline{\otimes} M$ is a unitary operator and $\omega : \mathcal{G} \to \mathbb{T}$ is a character satisfying $\omega(v)(v \otimes v)Z = Z\Delta(v)$ for all $v \in \mathcal{G}$.

Denoting by $\sigma : M \overline{\otimes} M \to M \overline{\otimes} M$ the flip map and using that $\sigma \circ \Delta = \Delta$, it follows that $Z\sigma(Z)^*$ commutes with all $v \otimes v$, $v \in \mathcal{G}$. By weak mixing, $Z\sigma(Z)^*$ is a multiple of 1. Using that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, we similarly find that $(Z \otimes 1)(\Delta \otimes \mathrm{id})(Z)$ is a multiple of $(1 \otimes Z)(\mathrm{id} \otimes \Delta)(Z)$. By [IPV10, Theorem 3.3], there exists a unitary $u \in M$ such that $Z = (u^* \otimes u^*)\Delta(u)$. But then,

$$\Delta(uvu^*) = \omega(v)\, uvu^* \otimes uvu^* \quad \text{for all} \;\; v \in \mathcal{G} \,.$$

This means that $uvu^* \in \mathbb{T}\Gamma$ for every $v \in \mathcal{G}$. □

5 Proof of Theorem B

As in [CIK13, Definition 2.7], we consider the class $\mathcal{C}_{\mathrm{rss}}$ of *relatively strongly solid* groups consisting of all nonamenable countable groups Γ such that for any tracial crossed product

$M = P \rtimes \Gamma$ and any von Neumann subalgebra $Q \subset pMp$ that is amenable relative to P, we have that either $Q \prec P$ or the normalizer $\mathcal{N}_{pMp}(Q)''$ stays amenable relative to P.

The class $\mathcal{C}_{\text{rss}}$ is quite large. Indeed, by [PV11, Theorem 1.6], all weakly amenable groups that admit a proper 1-cocycle into an orthogonal representation weakly contained in the regular representation belong to $\mathcal{C}_{\text{rss}}$. In particular, the free groups $\mathbb{F}_n$ with $2 \leq n \leq \infty$ belong to $\mathcal{C}_{\text{rss}}$ and more generally, all free products $\Lambda_1 * \Lambda_2$ of amenable groups Λ_1, Λ_2 with $|\Lambda_1| \geq 2$ and $|\Lambda_2| \geq 3$ belong to $\mathcal{C}_{\text{rss}}$. By [PV12, Theorem 1.4], all weakly amenable, nonamenable, bi-exact groups belong to $\mathcal{C}_{\text{rss}}$ and thus $\mathcal{C}_{\text{rss}}$ contains all nonelementary hyperbolic groups.

Theorem B is an immediate consequence of the more general Theorem 5.1 that we prove in this section. In order to make our statements entirely explicit, we call a group measure space (gms) decomposition of a tracial von Neumann algebra (M, τ) any pair (B, Λ) where $B \subset M$ is a maximal abelian von Neumann subalgebra and $\Lambda \subset \mathcal{U}(M)$ is a subgroup normalizing B such that $M = (B \cup \Lambda)''$ and $E_B(v) = 0$ for all $v \in \Lambda \setminus \{1\}$. This of course amounts to writing $M = B \rtimes \Lambda$ for some free and trace preserving action $\Lambda \curvearrowright (B, \tau)$.

We then say that two gms decompositions (B_i, Λ_i), $i = 0, 1$, of M are

- identical if $B_0 = B_1$ and $\mathbb{T}\Lambda_0 = \mathbb{T}\Lambda_1$;
- unitarily conjugate if there exists a unitary $u \in \mathcal{U}(M)$ such that $uB_0u^* = B_1$ and $u\mathbb{T}\Lambda_0u^* = \mathbb{T}\Lambda_1$;
- conjugate by an automorphism if there exists an automorphism $\theta \in \text{Aut}(M)$ such that $\theta(B_0) = B_1$ and $\theta(\mathbb{T}\Lambda_0) = \mathbb{T}\Lambda_1$.

Theorem 5.1. *Let Γ be a torsion free group in the class $\mathcal{C}_{\text{rss}}$. Let (A_0, τ_0) be any amenable tracial von Neumann algebra with $A_0 \neq \mathbb{C}1$ and $\beta : \Gamma \curvearrowright (A_0, \tau_0)$ any trace preserving action such that* $\operatorname{Ker}\beta$ *is a nontrivial subgroup of Γ. Define $(A, \tau) = (A_0, \tau_0)^\Gamma$ and denote by $\pi_k : A_0 \to A$ the embedding as the k'th tensor factor. Define the action $\sigma : \Gamma \times \Gamma \curvearrowright (A, \tau)$ given by $\sigma_{(g,h)}(\pi_k(a)) = \pi_{gkh^{-1}}(\beta_h(a))$ for all $g, k, h \in \Gamma$ and $a \in A_0$. Denote $M = A \rtimes (\Gamma \times \Gamma)$.*

Up to unitary conjugacy, all gms decompositions of M are given as $M = B \rtimes \Lambda$ with $B = B_0^\Gamma$ and $\Lambda = \Lambda_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ where $A_0 = B_0 \rtimes \Lambda_0$ is a gms decomposition of A_0 satisfying $\beta_g(B_0) = B_0$ and $\beta_g(\Lambda_0) = \Lambda_0$ for all $g \in \Gamma$.

Moreover, the gms decompositions of M associated with (B_0, Λ_0) and (B_1, Λ_1) are

(1) *unitarily conjugate iff (B_0, Λ_0) is identical to (B_1, Λ_1),*

(2) *conjugate by an automorphism of M iff there exists a trace preserving automorphism $\theta_0 : A_0 \to A_0$ and an automorphism $\varphi \in \text{Aut}(\Gamma)$ such that $\theta_0(B_0) = B_1$, $\theta_0(\mathbb{T}\Lambda_0) = \mathbb{T}\Lambda_1$ and $\theta_0 \circ \beta_g = \beta_{\varphi(g)} \circ \theta_0$ for all $g \in \Gamma$.*

Note that in Proposition 5.14 at the end of this section, we discuss when the Cartan subalgebras $B = B_0^\Gamma$ are unitarily conjugate, resp. conjugate by an automorphism of M.

Lemma 5.2. *Let Γ be a group in $\mathcal{C}_{\text{rss}}$ and $M = P \rtimes \Gamma$ any tracial crossed product. If $Q_1, Q_2 \subset pMp$ are commuting von Neumann subalgebras, then either $Q_1 \prec_M P$ or Q_2 is amenable relative to P.*

Proof. Assume that $Q_1 \not\prec_M P$. By [BO08, Corollary F.14], there exists a diffuse abelian von Neumann subalgebra $A \subset Q_1$ such that $A \not\prec_M P$. Because $\Gamma \in \mathcal{C}_{\text{rss}}$, we get that $\mathcal{N}_{pMp}(A)''$ is amenable relative to P. Since $Q_2 \subset \mathcal{N}_{pMp}(A)''$, also Q_2 is amenable relative to P. □

It also follows that for groups Γ in $\mathcal{C}_{\text{rss}}$, the centralizer $C_\Gamma(L)$ of an infinite subgroup $L < \Gamma$ is amenable. So, torsion free groups Γ in $\mathcal{C}_{\text{rss}}$ have the property that $C_\Gamma(g)$ is amenable for every $g \neq e$. As a consequence, torsion free groups Γ in $\mathcal{C}_{\text{rss}}$ are icc and even have the property that every nonamenable subgroup $L < \Gamma$ is *relatively icc* in the sense that $\{hgh^{-1} \mid h \in L\}$ is an infinite set for every $g \in \Gamma$, $g \neq e$. Finally note that torsion free groups Γ in $\mathcal{C}_{\text{rss}}$ have no nontrivial amenable normal subgroups. In particular, every nontrivial normal subgroup of Γ is relatively icc.

In the rest of this section, we prove Theorem 5.1. So, we fix a group Γ and an action $\Gamma \times \Gamma \curvearrowright A$ as in the formulation of the theorem. We put $M = A \rtimes (\Gamma \times \Gamma)$.

Lemma 5.3. *Let (N, τ) be a tracial factor and let $Q_1, Q_2 \subset N \overline{\otimes} M$ be commuting von Neumann subalgebras that are strongly nonamenable relative to $N \otimes 1$. Then $Q_1 \vee Q_2$ can be unitarily conjugated into $N \overline{\otimes} L(\Gamma \times \Gamma)$.*

Proof. Since A is amenable, we get that Q_1 and Q_2 are strongly nonamenable relative to $N \overline{\otimes} (A \rtimes L)$ whenever $L < \Gamma \times \Gamma$ is an amenable subgroup. For every $g \in \Gamma$, we denote by $\operatorname{Stab} g \subset \Gamma \times \Gamma$ the stabilizer of g under the left-right translation action $\Gamma \times \Gamma \curvearrowright \Gamma$. We also write $\operatorname{Stab}\{g, h\} = \operatorname{Stab} g \cap \operatorname{Stab} h$.

We start by proving that $Q_2 \not\prec N \overline{\otimes} (A \rtimes \operatorname{Stab} g)$ for all $g \in \Gamma$. Assume the contrary. Whenever $h \neq g$, the group $\operatorname{Stab}\{g, h\}$ is amenable so that by Proposition 2.2, $Q_2 \not\prec N \overline{\otimes} (A \rtimes \operatorname{Stab}\{g, h\})$. Also by Proposition 2.2, we can take projections $q \in Q_2$ and $p \in N \overline{\otimes} (A \rtimes \operatorname{Stab} g)$, a nonzero partial isometry $v \in q(N \overline{\otimes} M)p$ and a normal unital $*$-homomorphism

$$\theta : qQ_2q \to p(N \overline{\otimes} (A \rtimes \operatorname{Stab} g))p$$

such that $xv = v\theta(x)$ for all $x \in qQ_2q$ and such that, inside $N \overline{\otimes} (A \rtimes \operatorname{Stab} g)$, we have that $\theta(qQ_2q)$ is nonamenable relative to $N \overline{\otimes} A$ and we have that $\theta(qQ_2q) \not\prec N \overline{\otimes} (A \rtimes \operatorname{Stab}\{g, h\})$ whenever $h \neq g$.

Write $P := \theta(qQ_2q)' \cap p(N \overline{\otimes} M)p$. By Lemma 2.7, $P \subset p(N \overline{\otimes} (A \rtimes \operatorname{Stab} g))p$. In particular, $v^*v \in p(N \overline{\otimes} (A \rtimes \operatorname{Stab} g))p$ and we may assume that $v^*v = p$. Since $\operatorname{Stab} g \cong \Gamma$, we have $\operatorname{Stab} g \in \mathcal{C}_{\text{rss}}$ and Lemma 5.2 implies that $P \prec N \overline{\otimes} A$. Conjugating with v and writing $e = vv^* \in (Q_2' \cap (N \overline{\otimes} M))q$, we find that $e(Q_2' \cap (N \overline{\otimes} M))e \prec N \overline{\otimes} A$. Since $Q_1 \subset Q_2' \cap (N \overline{\otimes} M)$, it follows that $Q_1 \prec N \overline{\otimes} A$. By Proposition 2.2, this contradicts the strong nonamenability of Q_1 relative to $N \overline{\otimes} A$. So, we have proved that $Q_2 \not\prec N \overline{\otimes} (A \rtimes \operatorname{Stab} g)$ for all $g \in \Gamma$.

Since Q_1 is strongly nonamenable relative to $N \overline{\otimes} A$ and since $Q_2 \not\prec N \overline{\otimes} (A \rtimes \operatorname{Stab} g)$ for all $g \in \Gamma$, it follows from Theorem 2.6 that $v^* Q_2 v \subset N \overline{\otimes} L(\Gamma \times \Gamma)$ for some unitary $v \in N \overline{\otimes} M$. Since $v^* Q_2 v \not\prec N \overline{\otimes} L(\operatorname{Stab} g)$ for all $g \in \Gamma$, it follows from Lemma 2.7 that also $v^* Q_1 v \subset N \overline{\otimes} L(\Gamma \times \Gamma)$. This concludes the proof of the lemma. ∎

We now also fix a gms decomposition $M = B \rtimes \Lambda$. We view Λ as a subgroup of $\mathcal{U}(M)$. We denote by $\Delta : M \to M \overline{\otimes} M$ the associated dual coaction given by $\Delta(b) = b \otimes 1$ for all $b \in B$ and $\Delta(v) = v \otimes v$ for all $v \in \Lambda$.

Lemma 5.4. *Writing $Q_1 = L(\Gamma \times \{e\})$ and $Q_2 = L(\{e\} \times \Gamma)$, we have $\Delta(Q_2) \prec_{M \overline{\otimes} M} M \overline{\otimes} Q_i$ for either $i = 1$ or $i = 2$.*

Proof. By Proposition 2.4, $\Delta(Q_1)$ and $\Delta(Q_2)$ are strongly nonamenable relative to $M \otimes 1$. So by Lemma 5.3, we can take a unitary $v \in M \overline{\otimes} M$ such that

$$v^* \Delta(Q_1 \vee Q_2) v \subset M \overline{\otimes} L(\Gamma \times \Gamma) \; .$$

We therefore have the two commuting subalgebras $v^* \Delta(Q_1) v$ and $v^* \Delta(Q_2) v$ inside $M \overline{\otimes} L(\Gamma \times \Gamma)$. If $v^* \Delta(Q_1) v$ was amenable relative to both $M \overline{\otimes} Q_1$ and $M \overline{\otimes} Q_2$, then it would be amenable relative to $M \overline{\otimes} 1$ by [PV11, Proposition 2.7], which is not the case. Hence $v^*\Delta(Q_1)v$ is nonamenable relative to either $M \overline{\otimes} Q_1$ or $M \overline{\otimes} Q_2$. Assuming that $v^*\Delta(Q_1)v$ is nonamenable relative to $M \overline{\otimes} Q_1$, Lemma 5.2 implies that $\Delta(Q_2) \prec M \overline{\otimes} Q_1$. □

In the following three lemmas, we prove that Λ contains two commuting nonamenable subgroups $\Lambda_1, \Lambda_2 < \Lambda$. The method to produce such commuting subgroups is taken from [Io11] and our proofs of Lemmas 5.5, 5.6 and 5.7 are very similar to the proof of [Io11, Theorem 3.1]. The same method was also used in [CdSS15, Theorem 3.3]. For completeness, we provide all details.

Combining Theorem 3.1 and Lemma 5.4, we get the following.

Lemma 5.5. *Denote by $\mathcal{G}$ the family of all amenable subgroups of Λ. For either $i = 1$ or $i = 2$, there exists a finite set $x_1, \ldots, x_n \in M$ and a $\delta > 0$ such that the following holds: whenever $F \subset \Lambda$ is small relative to $\mathcal{G}$, we can find an element $v_F \in \Lambda - F$ such that*

$$\sum_{k,j=1}^{n} \|E_{Q_i}(x_k v_F x_j^*)\|_2^2 \geq \delta \; .$$

Lemma 5.6. *There exists a decreasing sequence of nonamenable subgroups $\Lambda_n < \Lambda$ such that $Q_i \prec_M B \rtimes (\bigcup_n C_\Lambda(\Lambda_n))$ for either $i = 1$ or $i = 2$, where $C_\Lambda(\Lambda_n)$ denotes the centralizer of Λ_n inside Λ.*

Proof. As in Lemma 5.5, we let $\mathcal{G}$ denote the family of all amenable subgroups of Λ. We denote by I the set of subsets of Λ that are small relative to $\mathcal{G}$. We order I by inclusion and choose a cofinal ultrafilter $\mathcal{U}$ on I. Consider the ultrapower von Neumann algebra $M^{\mathcal{U}}$ and the ultrapower group $\Lambda^{\mathcal{U}}$. Every $v = (v_F)_{F \in I} \in \Lambda^{\mathcal{U}}$ can be viewed as a unitary in $M^{\mathcal{U}}$.

Assume without loss of generality that $i = 1$ in Lemma 5.5 and denote by $v = (v_F)_{F \in I}$ the element of $\Lambda^{\mathcal{U}}$ that we found in Lemma 5.5. Denote by $K \subset L^2(M^{\mathcal{U}})$ the closed linear span of MvM and by P_K the orthogonal projection from $L^2(M^{\mathcal{U}})$ onto K. Put $\Sigma = \Lambda \cap v\Lambda v^{-1}$. We claim that $Q_2 \prec_M B \rtimes \Sigma$.

Assume the contrary. This means that we can find a sequence of unitaries $a_n \in \mathcal{U}(Q_2)$ such that $\|E_{B \rtimes \Sigma}(x a_n y)\|_2 \to 0$ for any $x, y \in M$. We prove that $\langle a_n \xi a_n^*, \eta \rangle \to 0$ as $n \to \infty$ for all $\xi, \eta \in K$. For this, it suffices to prove that $\langle a_n x v x' a_n^*, y v y' \rangle \to 0$ for all $x, x', y, y' \in M$. First, note that for all $z \in M$, we have $E_M(v^* z v) = E_M(v^* E_{B \rtimes \Sigma}(z) v)$ by definition of the subgroup Σ. Hence

$$\begin{aligned} \langle a_n x v x' a_n^*, y v y' \rangle &= \tau(E_M(v^* y^* a_n x v) x' a_n^* y'^*) \\ &= \tau(E_M(v^* E_{B \rtimes \Sigma}(y^* a_n x) v) x' a_n^* y'^*) \\ &\leq \|x'\| \|y'\| \|E_{B \rtimes \Sigma}(y^* a_n x)\|_2 \to 0 \end{aligned}$$

as wanted.

Next, Lemma 5.5 provides a finite set $L \subset M$ such that $\sum_{x,y \in L} \|E_{Q_1^{\mathcal{U}}}(x v y^*)\|_2^2 \neq 0$. In particular, we can take $x, y \in L$ such that $E_{Q_1^{\mathcal{U}}}(x v y^*) \neq 0$. Put $\xi = P_K(E_{Q_1^{\mathcal{U}}}(x v y^*))$. We claim that $\xi \neq 0$. Since $E_{Q_1^{\mathcal{U}}}(x v y^*) \neq 0$, we get that $\|x v y^* - E_{Q_1^{\mathcal{U}}}(x v y^*)\|_2 < \|x v y^*\|_2$. Since $x v y^* \in K$, it follows that $\|x v y^* - \xi\|_2 = \|P_K(x v y^* - E_{Q_1^{\mathcal{U}}}(x v y^*))\|_2 < \|x v y^*\|_2$. Hence $\xi \neq 0$.

Since K is an M-M-bimodule and since Q_1 commutes with Q_2, we have that $a\xi = \xi a$ for all $a \in Q_2$. In particular, $\langle a_n \xi a_n^*, \xi \rangle = \|\xi\|_2^2 > 0$ in contradiction with the fact that $\langle a_n \xi a_n^*, \xi \rangle \to 0$. This proves that $Q_2 \prec_M B \rtimes \Sigma$.

It remains to show that there exists a decreasing sequence of subgroups $\Lambda_n < \Lambda$ such that for all n we have $\Lambda_n \notin \mathcal{G}$, and such that $\Sigma = \bigcup_n C_\Lambda(\Lambda_n)$. For every $\mathcal{T} \subset I$, we denote by $\Lambda_{\mathcal{T}}$ the subgroup of Λ generated by $\{v_F v_{F'}^{-1} \mid F, F' \in \mathcal{T}\}$. An element $w \in \Lambda$ belongs to Σ if and only if there exists a $\mathcal{T} \in \mathcal{U}$ such that w commutes with $\Lambda_{\mathcal{T}}$. Enumerating $\Sigma = \{w_1, w_2, \ldots\}$, choose $\mathcal{S}_n \in \mathcal{U}$ such that w_n commutes with $\Lambda_{\mathcal{S}_n}$. Then put $\mathcal{T}_n := \mathcal{S}_1 \cap \ldots \cap \mathcal{S}_n$. Note that $\mathcal{T}_n \in \mathcal{U}$ and by construction, $\Sigma = \bigcup_n C_\Lambda(\Lambda_{\mathcal{T}_n})$. It remains to prove that $\Lambda_{\mathcal{T}} \notin \mathcal{G}$ for all $\mathcal{T} \in \mathcal{U}$.

Fix $\mathcal{T} \in \mathcal{U}$ and assume that $\Lambda_{\mathcal{T}} \in \mathcal{G}$. Fix an element $F' \in \mathcal{T}$. Then $\{v_F \mid F \in \mathcal{T}\} \subset \Lambda_{\mathcal{T}} v_{F'}$. So, $F_1 := \{v_F \mid F \in \mathcal{T}\}$ is small relative to $\mathcal{G}$. Define $\mathcal{T}' \subset I$ by $\mathcal{T}' = \{F \in I \mid F_1 \subset F\}$. Since $\mathcal{U}$ is a cofinal ultrafilter and $\mathcal{T} \in \mathcal{U}$, we get $\mathcal{T} \cap \mathcal{T}' \neq \emptyset$. So we can take $F \in \mathcal{T}$ with $F_1 \subset F$. Then, $v_F \in \Lambda - F \subset \Lambda - F_1$ but also $v_F \in F_1$. This being absurd, we have shown that $\Lambda_{\mathcal{T}} \notin \mathcal{G}$ for all $\mathcal{T} \in \mathcal{U}$. □

Lemma 5.7. *There exist two commuting nonamenable subgroups Λ_1 and Λ_2 inside Λ. Moreover, whenever $\Lambda_1, \Lambda_2 < \Lambda$ are commuting nonamenable subgroups, $L(\Lambda_1\Lambda_2)$ can be unitarily conjugated into $L(\Gamma \times \Gamma)$.*

Proof. From Lemma 5.6, we find a decreasing sequence of nonamenable subgroups $\Lambda_n < \Lambda$ such that $Q_i \prec_M B \rtimes (\bigcup_n C_\Lambda(\Lambda_n))$ for either $i = 1$ or $i = 2$. Since Q_i has no amenable direct summand, we get that the group $\bigcup_n C_\Lambda(\Lambda_n)$ is nonamenable. It follows that $C_\Lambda(\Lambda_n)$ is nonamenable for some $n \in \mathbb{N}$. Denote $\Lambda_1 := \Lambda_n$ and $\Lambda_2 := C_\Lambda(\Lambda_n)$.

When $\Lambda_1, \Lambda_2 < \Lambda$ are commuting nonamenable subgroups, it follows from Lemma 5.3 applied to $N = \mathbb{C}1$ that $L(\Lambda_1) \vee L(\Lambda_2)$ can be unitarily conjugated into $L(\Gamma \times \Gamma)$. □

From now on, we fix commuting nonamenable subgroups $\Lambda_1, \Lambda_2 < \Lambda$. By Lemma 5.7, after a unitary conjugacy, we may assume that $L(\Lambda_1\Lambda_2) \subset L(\Gamma \times \Gamma)$.

Lemma 5.8. *If $N \subset L(\Gamma\times\Gamma)$ is an amenable von Neumann subalgebra such that the normalizer $\mathcal{N}_{L(\Gamma\times\Gamma)}(N)''$ contains $L(\Lambda_1\Lambda_2)$, then N is atomic. Also, $L(\Lambda_1\Lambda_2)' \cap L(\Gamma \times \Gamma)$ is atomic.*

Proof. Using [Va10, Proposition 2.6], we find a projection q in the center of the normalizer of N such that $Nq \prec^f L(\Gamma) \otimes 1$ and $N(1 - q) \not\prec L(\Gamma) \otimes 1$.

Assume for contradiction that $q \neq 1$. Since $N(1 - q) \not\prec L(\Gamma) \otimes 1$ and since $\Gamma \in \mathcal{C}_{\text{rss}}$, it follows that $L(\Lambda_i)(1 - q)$ is amenable relative to $L(\Gamma) \otimes 1$ for both $i = 1, 2$. It then follows from [PV11, Proposition 2.7] that $L(\Lambda_1)(1 - q)$ is nonamenable relative to $1 \otimes L(\Gamma)$, hence $L(\Lambda_2)(1 - q) \prec 1 \otimes L(\Gamma)$ by Lemma 5.2. By Proposition 2.2, we get a nonzero projection $q_0 \leq 1 - q$ that commutes with $L(\Lambda_2)$ such that $L(\Lambda_2)q_0$ is amenable relative to $1 \otimes L(\Gamma)$. But since $L(\Lambda_2)q_0$ is also amenable relative to $L(\Gamma) \otimes 1$, it follows from [PV11, Proposition 2.7] that $L(\Lambda_2)q_0$ is amenable relative to $\mathbb{C}1$, hence a contradiction.

We conclude that $q = 1$ so that $N \prec^f L(\Gamma) \otimes 1$. By symmetry, we also get that $N \prec^f 1 \otimes L(\Gamma)$. Hence $N \prec^f \mathbb{C}1$, so that N is atomic.

To prove that $L(\Lambda_1\Lambda_2)' \cap L(\Gamma\times\Gamma)$ is atomic, it suffices to prove that every abelian von Neumann subalgebra $D \subset L(\Lambda_1\Lambda_2)' \cap L(\Gamma \times \Gamma)$ is atomic. But then D is amenable and its normalizer contains $L(\Lambda_1\Lambda_2)$, so that D is indeed atomic. □

The proof of the following lemma is essentially contained in the proof of [OP03, Proposition 12].

Lemma 5.9. *For every minimal projection $e \in L(\Lambda_1\Lambda_2)' \cap L(\Gamma \times \Gamma)$, there exist projections $p \in M_n(\mathbb{C}) \otimes L(\Gamma)$, $q \in L(\Gamma) \otimes M_m(\mathbb{C})$ and a partial isometry $u \in M_{n,1}(\mathbb{C}) \otimes L(\Gamma\times\Gamma) \otimes M_{m,1}(\mathbb{C})$ such that $u^*u = e$, $uu^* = p \otimes q$ and such that*

$$\begin{aligned} \text{either} \quad & uL(\Lambda_1)u^* \subset p(M_n(\mathbb{C}) \otimes L(\Gamma))p \otimes q \quad \text{and} \quad uL(\Lambda_2)u^* \subset p \otimes q(L(\Gamma) \otimes M_m(\mathbb{C}))q \ , \\ \text{or} \quad & uL(\Lambda_1)u^* \subset p \otimes q(L(\Gamma) \otimes M_m(\mathbb{C}))q \quad \text{and} \quad uL(\Lambda_2)u^* \subset p(M_n(\mathbb{C}) \otimes L(\Gamma))p \otimes q \ . \end{aligned}$$

Proof. By [PV11, Proposition 2.7], $L(\Lambda_2)e$ is nonamenable relative to either $L(\Gamma)\otimes 1$ or $1\otimes L(\Gamma)$. Assume that $L(\Lambda_2)e$ is nonamenable relative to $L(\Gamma)\otimes 1$. By Lemma 5.2, $L(\Lambda_1)e \prec L(\Gamma)\otimes 1$. Take a projection $p \in M_n(\mathbb{C})\otimes L(\Gamma)$, a nonzero partial isometry $v \in (p\otimes 1)(M_{n,1}(\mathbb{C})\otimes L(\Gamma\times\Gamma))e$ and a unital normal $*$-homomorphism $\theta : L(\Lambda_1) \to p(M_n(\mathbb{C})\otimes L(\Gamma))p$ such that

$$(\theta(x)\otimes 1)v = vx \quad \text{for all} \;\; x \in L(\Lambda_1)\; .$$

Since $\Gamma \in \mathcal{C}_{\text{rss}}$ and Λ_1 is nonamenable, the relative commutant $\theta(L(\Lambda_1))' \cap p(M_n(\mathbb{C})\otimes L(\Gamma))p$ is atomic. Cutting with a minimal projection, we may assume that this relative commutant equals $\mathbb{C}p$.

Write $P := L(\Lambda_1)' \cap L(\Gamma\times\Gamma)$ and note that $v^*v, e \in P$ with $v^*v \leq e$. Since $L(\Lambda_2) \subset P$, we have that $\mathcal{Z}(P) \subset L(\Lambda_1\Lambda_2)'\cap L(\Gamma\times\Gamma)$. It follows that $\mathcal{Z}(P)e = \mathbb{C}e$. So, ePe is a II$_1$ factor and we can take partial isometries $v_1,\ldots,v_m \in ePe$ with $v_i v_i^* \leq v^*v$ for all i and $\sum_{i=1}^m v_i^* v_i = e$. Define $u \in M_{n,1}(\mathbb{C})\otimes L(\Gamma\times\Gamma)\otimes M_{m,1}(\mathbb{C})$ given by $u = \sum_{i=1}^m v v_i \otimes e_{i1}$.

Since vPv^* commutes with $\theta(L(\Lambda_1))\otimes 1$, we have $vPv^* \subset p\otimes L(\Gamma)$ and we can define the normal $*$-homomorphism $\eta : v^*vPv^*v \to L(\Gamma)$ such that $vyv^* = p\otimes\eta(y)$ for all $y \in v^*vPv^*v$. By construction, $u^*u = e$ and $uu^* = p\otimes q$ where $q \in L(\Gamma)\otimes M_m(\mathbb{C})$ is the projection given by $q = \sum_{i=1}^m \eta(v_i v_i^*)\otimes e_{ii}$. Defining the $*$-homomorphism

$$\widetilde{\eta} : ePe \to q(L(\Gamma)\otimes M_m(\mathbb{C}))q : \widetilde{\eta}(y) = \sum_{i,j=1}^m \eta(v_i y v_j^*)\otimes e_{ij}$$

and using that $L(\Lambda_2)e \subset ePe$, we get that

$$uL(\Lambda_1)u^* = \theta(L(\Lambda_1))\otimes q \quad \text{and} \quad uL(\Lambda_2)u^* = p\otimes\widetilde{\eta}(L(\Lambda_2)e)\; .$$

This concludes the proof of the lemma. □

Recall from Section 4 the notion of height of an element in a group von Neumann algebra (here, $L(\Gamma\times\Gamma)$), as well as the height of a subgroup of $\mathcal{U}(L(\Gamma\times\Gamma))$. The proof of the following lemma is very similar to the proof of [Io10, Theorem 4.1].

Lemma 5.10. *For every projection $p \in L(\Lambda_1\Lambda_2)'\cap L(\Gamma\times\Gamma)$, we have that $h_{\Gamma\times\Gamma}(\Lambda_1\Lambda_2 p) > 0$.*

Proof. Fix a minimal projection $p \in L(\Lambda_1\Lambda_2)'\cap L(\Gamma\times\Gamma)$. It suffices to prove that $h_{\Gamma\times\Gamma}(\Lambda_1\Lambda_2 p) > 0$. Using the conjugacy of Lemma 5.9, we see that the heights of $\Lambda_1 p$ and $\Lambda_2 p$ do not interact, so that it suffices to prove that $h_{\Gamma\times\Gamma}(\Lambda_i p) > 0$ for $i = 1,2$. By symmetry, it is enough to prove this for $i = 1$.

Assume for contradiction that $h_{\Gamma\times\Gamma}(\Lambda_1 p) = 0$. Take a sequence $v_n \in \Lambda_1$ such that $h_{\Gamma\times\Gamma}(v_n p) \to 0$. For every finite subset $S \subset \Gamma\times\Gamma$, we denote by P_S the orthogonal projection of $L^2(M)$ onto the linear span of $L^2(A)u_g$, $g \in S$. We claim that for every sequence of unitaries $w_n \in L(\Gamma\times\Gamma)$, every $a \in M \ominus L(\Gamma\times\Gamma)$ and every finite subset $S \subset \Gamma\times\Gamma$, we have that

$$\lim_n \|P_S(pv_n a w_n)\|_2 = 0\; .$$

Since $P_S(x) = \sum_{g\in S} E_A(xu_g^*)u_g$, it suffices to prove that $\|E_A(pv_n a w_n)\|_2 \to 0$ for all $a \in M \ominus L(\Gamma\times\Gamma)$. Every such a can be approximated by a linear combination of elements of the form $a_0 u_g$ with $a_0 \in A \ominus \mathbb{C}1$ and $g \in \Gamma\times\Gamma$. So, we may assume that $a \in A\ominus\mathbb{C}1$. Such an element a can be approximated by a linear combination of elementary tensors, so that we may assume that $a = \bigotimes_{i\in\mathcal{G}} a_i$ for some finite nonempty subset $\mathcal{G} \subset \Gamma$ and elements $a_i \in A_0 \ominus \mathbb{C}1$ with $\|a\| \leq 1$. Note that $\sigma_g(a) \perp \sigma_h(a)$ whenever $g,h \in \Gamma\times\Gamma$ and $g\cdot\mathcal{G} \neq h\cdot\mathcal{G}$ (where we use the left right action of $\Gamma\times\Gamma$ on Γ).

Choose $\varepsilon > 0$. By Lemma 5.9, we can take a finite subset $F_0 \subset \Gamma$ such that, writing $F = \Gamma \times F_0 \cup F_0 \times \Gamma$, we have $\|pv - P_F(pv)\|_2 \leq \varepsilon$ for all $v \in \Lambda_1$. Then,

$$\|E_A(pv_n a w_n) - E_A(P_F(pv_n) a w_n)\|_2 \leq \varepsilon$$

for all n, so that in order to prove the claim, it suffices to prove that $\|E_A(P_F(pv_n)aw_n)\|_2 \to 0$. Put $\kappa = 2|F_0||\mathcal{G}|^2$. Note that for every $h \in \Gamma \times \Gamma$, the set $\{g \in F \mid g \cdot \mathcal{G} = h \cdot \mathcal{G}\}$ contains at most κ elements. Using the Fourier decomposition for elements in $L(\Gamma \times \Gamma)$, we have

$$E_A(P_F(pv_n)aw_n) = \sum_{g \in F} (pv_n)_g \, (w_n)_{g^{-1}} \, \sigma_g(a) \; .$$

Thus, for all $h \in \Gamma \times \Gamma$, we have

$$|\langle E_A(P_F(pv_n)aw_n), \sigma_h(a) \rangle| \leq \kappa \, h_{\Gamma \times \Gamma}(pv_n) \; .$$

But then, using the Cauchy-Schwarz inequality, we get that

$$\begin{aligned}
\|E_A(P_F(pv_n)aw_n)\|_2^2 &\leq \sum_{h \in F} |\langle E_A(P_F(pv_n)aw_n) \, , \, (pv_n)_h \, (w_n)_{h^{-1}} \, \sigma_h(a) \rangle| \\
&\leq \kappa \, h_{\Gamma \times \Gamma}(pv_n) \sum_{h \in \Gamma \times \Gamma} |(pv_n)_h| \, |(w_n)_{h^{-1}}| \\
&\leq \kappa \, h_{\Gamma \times \Gamma}(pv_n) \, \|pv_n\|_2 \, \|w_n\|_2 \leq \kappa \, h_{\Gamma \times \Gamma}(pv_n) \to 0 \; .
\end{aligned}$$

So, the claim is proved.

Put $\delta = \|p\|_2/4$. Because $\Gamma \in \mathcal{C}_{\text{rss}}$ and $B \subset M$ is a Cartan subalgebra, we have that $B \prec^f A$. By Lemma 2.1, we have $B \not\prec L(\Gamma \times \Gamma)$ and we can take a unitary $b \in \mathcal{U}(B)$ such that $\|E_{L(\Gamma \times \Gamma)}(b)\|_2 \leq \delta$. Since $B \prec^f A$, we can take a finite subset $S \subset \Gamma \times \Gamma$ such that $\|pd - P_S(pd)\|_2 \leq \delta$ for all $d \in \mathcal{U}(B)$. For every n, we have that $v_n b v_n^* \in \mathcal{U}(B)$. Therefore,

$$\|pv_n b v_n^* - P_S(pv_n b v_n^*)\|_2 \leq \delta$$

for all n. Since $\|E_{L(\Gamma \times \Gamma)}(b)\|_2 \leq \delta$, writing $b_1 = b - E_{L(\Gamma \times \Gamma)}(b)$, we get

$$\|P_S(pv_n b v_n^*) - P_S(pv_n b_1 v_n^*)\|_2 \leq \delta \; .$$

By the claim above, we can fix n large enough such that $\|P_S(pv_n b_1 v_n^*)\|_2 \leq \delta$. So, we have proved that

$$\|p\|_2 = \|pv_n b v_n^*\|_2 \leq 3\delta < \|p\|_2 \; ,$$

which is absurd. So, we have shown that $h_{\Gamma \times \Gamma}(p\Lambda_1) > 0$ and the lemma is proved. □

Lemma 5.11. *There exists a unitary $u \in L(\Gamma \times \Gamma)$ such that $u\Lambda_1\Lambda_2 u^* \subset \mathbb{T}(\Gamma \times \Gamma)$. Also, the unitary representation $\{\operatorname{Ad} v\}_{v \in \Lambda_1\Lambda_2}$ is weakly mixing on $L^2(M) \ominus \mathbb{C}1$.*

Proof. Write $\Lambda_0 = \Lambda_1\Lambda_2$. Denote the action of Λ on B by $\gamma_v(b) = vbv^*$ for all $v \in \Lambda$, $b \in B$. Define $K < \Lambda$ as the virtual centralizer of Λ_0 inside Λ, i.e. K consists of all $v \in \Lambda$ such that the set $\{wvw^{-1} \mid w \in \Lambda_0\}$ is finite. Define $B_0 \subset B$ as the von Neumann algebra generated by the unital $*$-algebra consisting of all $b \in B$ such that $\{\gamma_v(b) \mid v \in \Lambda_0\}$ spans a finite dimensional subspace of B. Note that B_0 is globally invariant under γ_v, $v \in \Lambda_0$. Viewing M as the crossed product $M = B \rtimes \Lambda$, we have by construction that the unitary representation $\{\operatorname{Ad} v\}_{v \in \Lambda_0}$ is weakly mixing on $L^2(M) \ominus L^2(B_0 \rtimes K)$.

For every $g \in \Gamma$, define $\operatorname{Stab} g$ as in the beginning of the proof of Lemma 5.3. We have $L(\Lambda_0) \subset L(\Gamma \times \Gamma)$ and $L(\Lambda_0) \not\prec L(\operatorname{Stab} g)$ for all $g \in \Gamma$. By Lemma 2.7, we have $B_0 \rtimes K \subset L(\Gamma \times \Gamma)$.

Since we can take decreasing sequences of finite index subgroups $\Lambda_{i,n} < \Lambda_i$, $i = 1,2$, such that $K = \bigcup_n C_\Lambda(\Lambda_{1,n}\Lambda_{2,n})$, it follows from Lemma 5.8 that $L(K)$ is contained in a hyperfinite von Neumann algebra. So, K is amenable and thus also $B_0 \rtimes K$ is amenable. Since $B_0 \rtimes K$ is normalized by Λ_0, it follows from Lemma 5.8 that $B_0 \rtimes K$ is atomic. So, K is a finite group and B_0 is atomic. We can then take a minimal projection $p \in B_0 \rtimes K$ and finite index subgroups $\Lambda_3 < \Lambda_1$ and $\Lambda_4 < \Lambda_2$ such that p commutes with $\Lambda_3\Lambda_4$.

Lemmas 5.8, 5.9 and 5.10 apply to the commuting nonamenable subgroups $\Lambda_3, \Lambda_4 < \Lambda$. So, by Lemma 5.10, we get that $h_{\Gamma\times\Gamma}(p\Lambda_3\Lambda_4) > 0$. By construction, the unitary representation $\{\mathrm{Ad}\, v\}_{v\in\Lambda_3\Lambda_4}$ is weakly mixing on $pL(\Gamma\times\Gamma)p \ominus \mathbb{C}p$. For every $g \in \Gamma\times\Gamma$ with $g \neq e$, the centralizer $C_{\Gamma\times\Gamma}(g)$ is either amenable or of the form $\Gamma\times L$ or $L\times\Gamma$ with $L < \Gamma$ amenable. Therefore, $L(\Lambda_3\Lambda_4) \not\prec L(C_{\Gamma\times\Gamma}(g))$ for all $g \neq e$. It then first follows from Theorem 4.1 that $p = 1$, so that we could have taken $\Lambda_3 = \Lambda_1$ and $\Lambda_4 = \Lambda_2$, and then also that there exists a unitary $u \in L(\Gamma\times\Gamma)$ such that $u\Lambda_1\Lambda_2u^* \subset \mathbb{T}(\Gamma\times\Gamma)$.

Since we also proved that $B_0 \rtimes K = \mathbb{C}1$, it follows as well that the unitary representation $\{\mathrm{Ad}\, v\}_{v\in\Lambda_1\Lambda_2}$ is weakly mixing on $L^2(M) \ominus \mathbb{C}1$. □

Lemma 5.12. *Whenever $\Lambda_2 \subset \mathbb{T}(\{e\}\times\Gamma)$ is a nonamenable subgroup, we have $M \cap \Lambda_2' = L(\Gamma)\otimes 1$.*

Proof. Define $\Gamma_2 < \Gamma$ such that $\mathbb{T}\Lambda_2 = \mathbb{T}(\{e\}\times\Gamma_2)$. Then Γ_2 is nonamenable and $M \cap \Lambda_2' = M \cap L(\{e\}\times\Gamma_2)'$. Since $\Gamma_2 < \Gamma$ is relatively icc, we have $M \cap L(\{e\}\times\Gamma_2)' \subset A \rtimes (\Gamma\times\{e\})$. Since the action $\{e\}\times\Gamma \curvearrowright A$ is mixing, it follows that $M \cap L(\{e\}\times\Gamma_2)' \subset L(\Gamma\times\{e\})$. So, $M\cap\Lambda_2' \subset L(\Gamma)\otimes 1$ and the converse inclusion is obvious. □

Lemma 5.13. *There exist commuting subgroups $H_1, H_2 < \Lambda$ and a unitary $u \in M$ such that $\Lambda_i < H_i$ for $i = 1,2$ and $u\mathbb{T}H_1H_2u^* = \mathbb{T}(\Gamma\times\Gamma)$.*

Proof. By Lemma 5.11, after a unitary conjugacy, we may assume that $\Lambda_1, \Lambda_2 < \Lambda$ are commuting nonamenable subgroups with $\Lambda_1\Lambda_2 \subset \mathbb{T}(\Gamma\times\Gamma)$. Since Γ is torsion free and belongs to $\mathcal{C}_{\mathrm{rss}}$, we have that $C_\Gamma(g)$ is amenable for every $g \neq e$. Therefore, after exchanging Λ_1 and Λ_2 if needed, we have $\Lambda_1 \subset \mathbb{T}(\Gamma\times\{e\})$ and $\Lambda_2 \subset \mathbb{T}(\{e\}\times\Gamma)$.

Denote by $\{\gamma_v\}_{v\in\Lambda}$ the action of Λ on B. Define $H_1 < \Lambda$ as the virtual centralizer of Λ_2 inside Λ. So, H_1 consists of all $v \in \Lambda$ that commute with a finite index subgroup of Λ_2. Similarly, define B_1 as the von Neumann algebra generated by the $*$-algebra consisting of all $b \in B$ such that $\gamma_v(b) = b$ for all v in a finite index subgroup of Λ_2. Since finite index subgroups of Λ_2 are nonamenable, it follows from Lemma 5.12 that $B_1 \rtimes H_1 \subset L(\Gamma)\otimes 1$. We also find that

$$L(\Gamma)\otimes 1 \subset L(\Lambda_2)' \cap (B\rtimes\Lambda) \subset B_1 \rtimes H_1 \; .$$

So, $B_1 \rtimes H_1 = L(\Gamma)\otimes 1$. In particular, the subgroups $H_1, \Lambda_2 < \Lambda$ commute. Because $\Gamma \in \mathcal{C}_{\mathrm{rss}}$ and $B_1 \subset L(\Gamma)\otimes 1$ is normalized by H_1, it follows that B_1 is atomic. Since $\Lambda_1 < H_1$, the unitaries $v \in \Lambda_1\Lambda_2$ normalize B_1. By Lemma 5.11, they induce a weakly mixing action on B_1. Since B_1 is atomic, this forces $B_1 = \mathbb{C}1$. We conclude that $L(H_1) = L(\Gamma)\otimes 1$.

We now apply Lemmas 5.8, 5.9 and 5.10 to the commuting nonamenable subgroups $H_1, \Lambda_2 < \Lambda$. We conclude that $h_\Gamma(H_1) > 0$. Since $L(H_1) = L(\Gamma)\otimes 1$, the group H_1 is icc. So, the action $\{\mathrm{Ad}\, v\}_{v\in H_1}$ on $L(\Gamma)\ominus\mathbb{C}1$ is weakly mixing. Since for $g \neq e$, the group $C_\Gamma(g)$ is amenable, also $L(H_1) \not\prec L(C_\Gamma(g))$. So, by Theorem 4.1, there exists a unitary $u_1 \in L(\Gamma)$ such that $(u_1\otimes 1)H_1(u_1^*\otimes 1) = \mathbb{T}(\Gamma\times\{e\})$.

Applying the same reasoning as above to the virtual centralizer of H_1 inside Λ, we find a subgroup $H_2 < \Lambda$, containing Λ_2 and commuting with H_1, and we find a unitary $u_2 \in L(\Gamma)$ such that $(1 \otimes u_2)H_2(1 \otimes u_2^*) \subset \mathbb{T}(\{e\} \rtimes \Gamma)$. So, we get that

$$(u_1 \otimes u_2)H_1H_2(u_1^* \otimes u_2^*) = \mathbb{T}(\Gamma \times \Gamma) .$$

$\square$

Finally, we are ready to prove Theorem 5.1. As mentioned above, Theorem B is a direct consequence of Theorem 5.1.

Proof of Theorem 5.1. First assume that (B_0, Λ_0) is a gms decomposition of A_0 satisfying $\beta_g(B_0) = B_0$ and $\beta_g(\Lambda_0) = \Lambda_0$ for all $g \in \Gamma$. Then, $\{\beta_g\}_{g\in\Gamma}$ defines an action of Γ by automorphisms of the group Λ_0. We can co-induce this to the action of $\Gamma \times \Gamma$ by automorphisms of the direct sum group $\Lambda_0^{(\Gamma)}$ given by $(g, h) \cdot \pi_k(v) = \pi_{gkh^{-1}}(\beta_h(v))$ for all $g, h, k \in \Gamma$, $v \in \Lambda_0$, where $\pi_k : \Lambda_0 \to \Lambda_0^{(\Gamma)}$ denotes the embedding as the k'th direct summand. Putting $B = B_0^\Lambda$ and $\Lambda := \Lambda_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, we have found the crossed product decomposition $M = B \rtimes \Lambda$. It is easy to check that $B \subset M$ is maximal abelian, so that (B, Λ) is indeed a gms decomposition of M.

Conversely, assume that (B, Λ) is an arbitrary gms decomposition of M. By Lemma 5.13 and after a unitary conjugacy, we have $\Gamma \times \Gamma \subset \mathbb{T}\Lambda$. Denoting by $\Delta : M \to M \mathbin{\overline{\otimes}} M$ the dual coaction associated with (B, Λ) and given by $\Delta(b) = b \otimes 1$ for all $b \in B$ and $\Delta(v) = v \otimes v$ for all $v \in \Lambda$, this means that $\Delta(u_{(g,h)})$ is a multiple of $u_{(g,h)} \otimes u_{(g,h)}$ for all $(g, h) \in \Gamma \times \Gamma$.

Denote $A_{0,e} := \pi_e(A_0) \subset A$ and observe that $A_{0,e}$ commutes with all $u_{(g,g)}$, $g \in \operatorname{Ker} \beta$. Then, $\Delta(A_{0,e})$ commutes with all $u_{(g,g)} \otimes u_{(g,g)}$, $g \in \operatorname{Ker} \beta$. Since Γ is a torsion free group in $\mathcal{C}_{\text{rss}}$, the nontrivial normal subgroup $\operatorname{Ker} \beta < \Gamma$ must be nonamenable and thus relatively icc. It follows that the unitary representation $\{\operatorname{Ad} u_{(g,g)}\}_{g\in\operatorname{Ker}\beta}$ is weakly mixing on $L^2(M) \ominus L^2(A_{0,e})$. This implies that $\Delta(A_{0,e}) \subset A_{0,e} \mathbin{\overline{\otimes}} A_{0,e}$.

By Lemma 2.3, we get a crossed product decomposition $A_0 = B_0 \rtimes \Lambda_0$ such that $\pi_e(B_0) = B \cap A_{0,e}$ and $\pi_e(\Lambda_0) = \Lambda \cap A_{0,e}$. For every $g \in \Gamma$, we have that $u_{(g,g)} \in \mathbb{T}\Lambda$. So, $u_{(g,g)}$ normalizes both B and $A_{0,e}$, so that $\beta_g(B_0) = B_0$. Also, $u_{(g,g)}$ normalizes both Λ and $A_{0,e}$, so that $\beta_g(\Lambda_0) = \Lambda_0$. For every $g \in \Gamma$, we have that $u_{(g,e)} \in \mathbb{T}\Lambda$ so that $u_{(g,e)}$ normalizes B and Λ. It follows that $\pi_g(B_0) \subset B$ and $\pi_g(\Lambda_0) \subset \Lambda$ for all $g \in \Gamma$. We conclude that

$$B_0^\Gamma \subset B \quad \text{and} \quad \Lambda_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma) \subset \mathbb{T}\Lambda . \tag{5.1}$$

Since A_0 is generated by B_0 and Λ_0, we get that M is generated by B_0^Γ and $\Lambda_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$. Since M is also the crossed product of B and Λ, it follows from (5.1) that $B_0^\Gamma = B$ and $\mathbb{T}\Lambda_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma) = \mathbb{T}\Lambda$. In particular, $B_0 \subset A_0$ must be maximal abelian. So, (B_0, Λ_0) is a gms decomposition of A_0 that is $\{\beta_g\}_{g\in\Gamma}$-invariant, while the gms decomposition (B, Λ) of M is unitarily conjugate to the gms decomposition associated with (B_0, Λ_0).

It remains to prove statements (1) and (2). Take $\{\beta_g\}_{g\in\Gamma}$-invariant gms decompositions (B_0, Λ_0) and (B_1, Λ_1) of A_0. Denote by (B, Λ) and (B', Λ') the associated gms decompositions of M.

To prove (1), assume that $u \in M$ is a unitary satisfying $uBu^* = B'$ and $u\mathbb{T}\Lambda u^* = \mathbb{T}\Lambda'$. It follows that for all $g \in \Gamma \times \Gamma$, we have $uu_gu^* \in \mathcal{U}(A)u_{\varphi(g)}$ where $\varphi \in \operatorname{Aut}(\Gamma \times \Gamma)$. Write $u = \sum_{h\in\Gamma\times\Gamma} a_h u_h$ with $a_h \in A$ for the Fourier decomposition of u. It follows that $\{\varphi(g)^{-1}hg \mid g \in \Gamma \times \Gamma\}$ is a finite set whenever $a_h \neq 0$. Since $\Gamma \times \Gamma$ is icc, it follows that a_h can only be nonzero for one $h \in \Gamma \times \Gamma$. So u is of the form $u = a_h u_h$. Since u_h normalizes both B and Λ, we may replace u with uu_h^* so that $u \in \mathcal{U}(A)$.

For each $g \in \Gamma$, we define $E_g\colon A \to A_0$ by $E_g(x) = \pi_g^{-1}(E_{\pi_g(A_0)}(x))$, $x \in A$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in Γ that tends to infinity, and let $b \in B_0$. Since $(\pi_{g_n}(b))_{n\in\mathbb{N}}$ is an asymptotically central sequence in A, we get that

$$B_1 \ni E_{g_n}(u\pi_{g_n}(b)u^*) \to b \;,$$

hence $B_0 \subset B_1$. By symmetry, it follows that $B_0 = B_1$. Similarly, we see that $\mathbb{T}\Lambda_0 = \mathbb{T}\Lambda_1$ so we conclude that (B_0, Λ_0) and (B_1, Λ_1) are identical gms decompositions of A_0.

To prove (2), assume that $\theta \in \operatorname{Aut}(M)$ is an automorphism satisfying $\theta(B) = B'$ and $\theta(\mathbb{T}\Lambda) = \mathbb{T}\Lambda'$. Define the commuting subgroups $\Lambda_1, \Lambda_2 < \Lambda'$ such that $\theta(\mathbb{T}(\Gamma \times \{e\})) = \mathbb{T}\Lambda_1$ and $\theta(\mathbb{T}(\{e\} \times \Gamma)) = \mathbb{T}\Lambda_2$. Applying Lemma 5.13 to the gms decomposition (B', Λ') of M and these commuting subgroups $\Lambda_1, \Lambda_2 < \Lambda'$, we find commuting subgroups $H_1, H_2 < \Lambda'$ and a unitary $u \in M$ such that $\Lambda_i < H_i$ for $i = 1, 2$ and $u\mathbb{T}H_1H_2u^* = \mathbb{T}(\Gamma \times \Gamma)$. Since $\Gamma \times \{e\}$ and $\{e\} \times \Gamma$ are each other's centralizer inside Λ and since $\theta(\mathbb{T}\Lambda) = \mathbb{T}\Lambda'$, we must have that $\Lambda_i = H_i$ for $i = 1, 2$.

Writing $\theta_1 = \operatorname{Ad} u \circ \theta$, we have proved that $\theta_1(\mathbb{T}(\Gamma \times \Gamma)) = \mathbb{T}(\Gamma \times \Gamma)$. This equality induces an automorphism of $\Gamma \times \Gamma$. Since Γ is a torsion free group in $\mathcal{C}_{\text{rss}}$, all automorphisms of $\Gamma \times \Gamma$ are either of the form $(g,h) \mapsto (\varphi_1(g), \varphi_2(h))$ or of the form $(g,h) \mapsto (\varphi_1(h), \varphi_2(g))$ for some automorphisms $\varphi_i \in \operatorname{Aut}(\Gamma)$. The formulas $\zeta(u_{(g,h)}) = u_{(h,g)}$ and $\zeta(\pi_k(a)) = \pi_{k^{-1}}(\beta_k(a))$ for all $g, h, k \in \Gamma$ and $a \in A_0$ define an automorphism $\zeta \in \operatorname{Aut}(M)$ satisfying $\zeta(B') = B'$ and $\zeta(\Lambda') = \Lambda'$. So composing θ with ζ if necessary, we may assume that we have $\varphi_1, \varphi_2 \in \operatorname{Aut}(\Gamma)$ such that $\theta_1(u_{(g,h)}) \in \mathbb{T}u_{(\varphi_1(g),\varphi_2(h))}$ for all $g, h \in \Gamma$. We still have that the gms decompositions $(\theta_1(B), \theta_1(\Lambda))$ and (B', Λ') of M are unitarily conjugate.

Because $u_{(g,g)}$ commutes with $\pi_e(A_0)$ for all $g \in \operatorname{Ker}\beta$, the unitary representation on $L^2(M) \ominus \mathbb{C}1$ given by $\{\operatorname{Ad} u_{(\varphi_1(g),\varphi_2(g))}\}_{g\in\operatorname{Ker}\beta}$ is not weakly mixing. There thus exists a $k \in \Gamma$ such that $\varphi_1(g)k = k\varphi_2(g)$ for all g in a finite index subgroup of $\operatorname{Ker}\beta$. So after replacing θ by $(\operatorname{Ad} u_{(e,k)}) \circ \theta$, which globally preserves B' and Λ', we may assume that $\varphi_1(g) = \varphi_2(g)$ for all g in a finite index subgroup of $\operatorname{Ker}\beta$. Since $\operatorname{Ker}\beta < \Gamma$ is relatively icc, this implies that $\varphi_1(g) = \varphi_2(g)$ for all $g \in \operatorname{Ker}\beta$. Since $\operatorname{Ker}\beta$ is a normal subgroup of Γ, it follows that $\varphi_1(k)\varphi_2(k)^{-1}$ commutes with $\varphi_1(g)$ for all $k \in \Gamma$ and $g \in \operatorname{Ker}\beta$. So, $\varphi_1 = \varphi_2$ and we denote this automorphism as φ.

Taking the commutant of the unitaries $u_{(g,g)}$, $g \in \operatorname{Ker}\beta$, it follows that $\theta_1(\pi_e(A_0)) = \pi_e(A_0)$. We define the automorphism $\theta_0 \in \operatorname{Aut}(A_0)$ such that $\theta_1 \circ \pi_e = \pi_e \circ \theta_0$. Since $\theta_1(u_{(g,g)}) \in \mathbb{T}u_{(\varphi(g),\varphi(g))}$, we get that $\theta_0 \circ \beta_g = \beta_{\varphi(g)} \circ \theta_0$ for all $g \in \Gamma$. It follows that $(\theta_0(B_0), \theta_0(\Lambda_0))$ is a $\{\beta_g\}_{g\in\Gamma}$-invariant gms decomposition of A_0. The associated gms decomposition of M is $(\theta_1(B), \theta_1(\Lambda))$. This gms decomposition of M is unitarily conjugate with the gms decomposition (B', Λ'). It then follows from (1) that $\theta_0(B_0) = B_1$ and $\theta_0(\mathbb{T}\Lambda_0) = \mathbb{T}\Lambda_1$. $\square$

Proposition 5.14. *Under the same hypotheses and with the same notations as in Theorem 5.1, if (B_0, Λ_0) and (B_1, Λ_1) are $\{\beta_g\}_{g\in\Gamma}$-invariant gms decompositions of A_0, then the associated Cartan subalgebras of M given by B_0^Γ and B_1^Γ are*

(1) *unitarily conjugate iff $B_0 = B_1$;*

(2) *conjugate by an automorphism of M iff there exists a trace preserving automorphism $\theta_0 : A_0 \to A_0$ and an automorphism $\varphi \in \operatorname{Aut}(\Gamma)$ such that $\theta_0(B_0) = B_1$ and $\theta_0 \circ \beta_g = \beta_{\varphi(g)} \circ \theta_0$ for all $g \in \Gamma$.*

Proof. To prove (1), it suffices to prove that $B_0^\Gamma \not\prec B_1^\Gamma$ if $B_0 \neq B_1$. Take a unitary $u \in \mathcal{U}(B_0)$ such that $u \notin B_1$. Then $\|E_{B_1}(u)\|_2 < 1$. Let $\{g_1, g_2, \ldots\}$ be an enumeration of Γ and define the sequence of unitaries $(w_n) \subset \mathcal{U}(B_0^\Gamma)$ by $w_n = \pi_{g_{n+1}}(u)\,\pi_{g_{n+2}}(u)\,\cdots\,\pi_{g_{2n}}(u)$. Then $\|E_{B_1^\Gamma}(xw_ny)\|_2 \to 0$ for all $x, y \in M$ so that $B_0^\Gamma \not\prec B_1^\Gamma$.

To prove (2), denote by (B,Λ) and (B',Λ') the gms decompositions of M associated with (B_0,Λ_0) and (B_1,Λ_1). Assume that $\theta \in \mathrm{Aut}(M)$ satisfies $\theta(B) = B'$. Then, $(B',\theta(\Lambda))$ is a gms decomposition of M. By Theorem 5.1, $(B',\theta(\Lambda))$ is unitarily conjugate with the gms decomposition associated with a $\{\beta_g\}_{g\in\Gamma}$-invariant gms decomposition (B_2,Λ_2) of A_0. By (1), we must have $B_2 = B_1$. So the gms decompositions associated with (B_0,Λ_0) and (B_1,Λ_2) are conjugate by an automorphism of M. By Theorem 5.1(2) there exists an automorphism $\theta_0 \in \mathrm{Aut}(A_0)$ as in (2). □

6 Examples of II_1 factors with a prescribed number of group measure space decompositions

For every amenable tracial von Neumann algebra (A_0,τ_0) and for every trace preserving action of $\Gamma = \mathbb{F}_\infty$ on (A_0,τ_0) with nontrivial kernel, Theorem 5.1 gives a complete description of all gms decompositions of the II_1 factor $M = A_0^\Gamma \rtimes (\Gamma \times \Gamma)$ in terms of the Γ-invariant gms decompositions of A_0.

In this section, we construct a family of examples where these Γ-invariant gms decompositions of A_0 can be explicitly determined. In particular, this gives a proof of Theorem A. We will construct A_0 of the form $A_0 = L^\infty(K) \rtimes H_1$ where H_1 is a countable abelian group and $H_1 \hookrightarrow K$ is an embedding of H_1 as a dense subgroup of the compact second countable group K. Note that we can equally view K as $\widehat{H_2}$ where H_2 is a countable abelian group and the embedding $H_1 \hookrightarrow \widehat{H_2}$ is given by a bicharacter $\Omega : H_1 \times H_2 \to \mathbb{T}$ that is nondegenerate: if $g \in H_1$ and $\Omega(g,h) = 1$ for all $h \in H_2$, then $g = e$; if $h \in H_2$ and $\Omega(g,h) = 1$ for all $g \in H_1$, then $h = e$.

We can then view $L^\infty(\widehat{H_2}) \rtimes H_1$ as being generated by the group von Neumann algebras $L(H_1)$ and $L(H_2)$, with canonical unitaries $\{u_g\}_{g\in H_1}$ and $\{u_h\}_{h\in H_2}$, satisfying $u_g u_h = \Omega(g,h) u_h u_g$ for all $g \in H_1$, $h \in H_2$.

We call a direct sum decomposition $H_1 = S_1 \oplus T_1$ *admissible* if the closures of S_1, T_1 in $\widehat{H_2}$ give a direct sum decomposition of $\widehat{H_2}$. This is equivalent to saying that $H_2 = S_2 \oplus T_2$ in such a way that $S_1 = S_2^\perp$, $S_2 = S_1^\perp$, $T_2 = T_1^\perp$ and $T_1 = T_2^\perp$, where the "orthogonal complement" is defined w.r.t. Ω.

Proposition 6.1. *Let L_1, L_2 be torsion free abelian groups and $L_1 \hookrightarrow \widehat{L_2}$ a dense embedding. Put $\Gamma_0 = \mathrm{SL}(3,\mathbb{Z})$ and $H_i = L_i^3$. Consider the natural action of Γ_0 on the direct sum embedding $H_1 \hookrightarrow \widehat{H_2}$, defining the trace preserving action $\{\beta_g\}_{g\in\Gamma_0}$ of Γ_0 on $A_0 = L^\infty(\widehat{H_2}) \rtimes H_1$.*

Whenever $L_1 = P_1 \oplus Q_1$ is an admissible direct sum decomposition with corresponding $L_2 = P_2 \oplus Q_2$, put $S_i = P_i^3$, $T_i = Q_i^3$ and define $B_0 = L(S_1) \vee L(S_2)$, $\Lambda_0 = T_1 T_2$.

Then (B_0,Λ_0) is a $\{\beta_g\}_{g\in\Gamma_0}$-invariant gms decomposition of A_0. Every $\{\beta_g\}_{g\in\Gamma_0}$-invariant gms decomposition of A_0 is of this form for a unique admissible direct sum decomposition $L_1 = P_1 \oplus Q_1$.

Proof. Let (B_0,Λ_0) be a $\{\beta_g\}_{g\in\Gamma_0}$-invariant gms decomposition of A_0. Define the subgroup $\Gamma_1 < \Gamma_0$ as

$$\Gamma_1 = \Gamma_0 \cap \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

We also put $H_1^{(1)} = L_1 \oplus 0 \oplus 0$. Because L_1 is torsion free, the following holds.

- $a \cdot g = g$ for all $a \in \Gamma_1$ and $g \in H_1^{(1)}$.

- $\Gamma_1 \cdot g$ is infinite for all $g \in H_1 \setminus H_1^{(1)}$.
- $\Gamma_1^T \cdot h$ is infinite for all $h \in H_2 \setminus \{0\}$, where Γ_1^T denotes the transpose of Γ_1.

From these observations, it follows that $L(H_1^{(1)})$ is equal to the algebra of Γ_1-invariant elements in A_0 and that $L(H_1^{(1)})$ is also equal to the algebra of elements in A_0 that are fixed by some finite index subgroup of Γ_1. Since both B_0 and Λ_0 are globally Γ_0-invariant, it follows that $L(H_1^{(1)}) = B_0^{(1)} \rtimes \Lambda_0^{(1)}$ for some von Neumann subalgebra $B_0^{(1)}$ and subgroup $\Lambda_0^{(1)} < \Lambda_0$.

We similarly consider $H_1^{(2)} = 0 \oplus L_1 \oplus 0$ and $H_1^{(3)} = 0 \oplus 0 \oplus L_1$. We conclude that $L(H_1^{(i)}) = B_0^{(i)} \rtimes \Lambda_0^{(i)}$ for all $i = 1, 2, 3$. The subgroups $H_1^{(1)}, H_1^{(2)}$ and $H_1^{(3)}$ generate H_1 and H_1 is abelian. So, "everything" commutes and we conclude that $L(H_1) = B_1 \rtimes \Lambda_1$ for some von Neumann subalgebra $B_1 \subset B_0$ and subgroup $\Lambda_1 < \Lambda_0$.

A similar reasoning applies to $L^\infty(\widehat{H_2}) = L(H_2)$ and we get that $L(H_2) = B_2 \rtimes \Lambda_2$ for $B_2 \subset B$ and $\Lambda_2 < \Lambda_0$.

Since $L(H_1)L(H_2)$ is $\|\cdot\|_2$-dense in A_0 and $L(H_1) \cap L(H_2) = \mathbb{C}1$, we get that $\Lambda_1\Lambda_2 = \Lambda_0 = \Lambda_2\Lambda_1$ and $\Lambda_1 \cap \Lambda_2 = \{e\}$. It then follows that for all $b_i \in B_i$ and $s_i \in \Lambda_i$, $i = 1, 2$, we have that $E_{B_0}(b_1 v_{s_1} b_2 v_{s_2})$ equals zero unless $s_1 = e$ and $s_2 = e$, in which case, we get $b_1 b_2$. We conclude that $B_1 B_2$ is $\|\cdot\|_2$-dense in B_0.

For every $x \in L(H_i)$ and $g \in H_i$, we denote by $(x)_g = \tau(x u_g^*)$ the g-th Fourier coefficient of x. Comparing Fourier decompositions, we get for all $x_i \in L(H_i)$ that

$$x_1 x_2 = x_2 x_1 \;\text{ iff }\; \Omega(g, h) = 1 \;\text{ whenever }\; g \in H_1, h \in H_2, (x_1)_g \neq 0, (x_2)_h \neq 0 \,. \tag{6.1}$$

Since $B_i \subset L(H_i)$ and since B_1, B_2 commute, we obtain from (6.1) subgroups $S_i \subset H_i$ such that $\Omega(g, h) = 1$ for all $g \in S_1$, $h \in S_2$ and such that $B_i \subset L(S_i)$. Since $B_1 B_2$ is dense in B_0, it follows that $B_0 \subset L(S_1) \vee L(S_2)$. Since $L(S_1) \vee L(S_2)$ is abelian and B_0 is maximal abelian, we conclude that $B_0 = L(S_1) \vee L(S_2)$. Thus, $B_i = L(S_i)$ for $i = 1, 2$. When $g \in S_2^\perp$, the unitary u_g commutes with $L(S_2)$, but also with $L(S_1)$ because $L(S_1) = B_1 \subset L(H_1)$ and $L(H_1)$ is abelian. Since B_0 is maximal abelian, we get that $g \in S_1$. So, $S_1 = S_2^\perp$ and similarly $S_2 = S_1^\perp$.

The next step of the proof is to show that Λ_0 is abelian, i.e. that Λ_1 and Λ_2 are commuting subgroups of Λ_0. Put $T_i = H_i / S_i$. Since $S_1 = S_2^\perp$ and $S_2 = S_1^\perp$, we have the canonical dense embeddings $T_1 \hookrightarrow \widehat{S_2}$ and $T_2 \hookrightarrow \widehat{S_1}$. Viewing $L^\infty(\widehat{S_1} \times \widehat{S_2}) = L(S_1) \vee L(S_2)$ as a Cartan subalgebra of A_0, the associated equivalence relation is given by the orbits of the action

$$T_1 \times T_2 \curvearrowright \widehat{S_1} \times \widehat{S_2} : (g, h) \cdot (y, z) = (h \cdot y, g \cdot z) \,,$$

where the actions on the right, namely $T_1 \curvearrowright \widehat{S_2}$ and $T_2 \curvearrowright \widehat{S_1}$, are given by translation. But viewing $B_0 = L(S_1) \vee L(S_2)$, the same equivalence relation is given by the orbits of the action $\Lambda_0 \curvearrowright \widehat{S_1} \times \widehat{S_2}$. We denote by $\omega : (T_1 \times T_2) \times (\widehat{S_1} \times \widehat{S_2}) \to \Lambda_0$ the associated 1-cocycle. By construction, for all $g \in H_1$, $h \in H_2$ and $s \in \Lambda_0$, the support of $E_B(v_s^* u_g u_h)$ is the projection in $L^\infty(\widehat{S_1} \times \widehat{S_2})$ given by the set

$$\{(y, z) \in \widehat{S_1} \times \widehat{S_2} \mid \omega((gS_1, hS_2), (y, z)) = s\} \,.$$

Since $L(H_1) = B_1 \rtimes \Lambda_1$, for all $g \in T_1$, the map $(y, z) \mapsto \omega((g, e), (y, z))$ only depends on the first variable and takes values in Λ_1 a.e. Reasoning similarly for $h \in T_2$, we find $\omega_i : T_i \times \widehat{S_i} \to \Lambda_i$ such that

$$\omega((g, e), (y, z)) = \omega_1(g, y) \quad \text{and} \quad \omega((e, h), (y, z)) = \omega_2(h, z) \quad \text{a.e.}$$

Writing $(g, h) = (g, e)(e, h)$ and $(g, h) = (e, h)(g, e)$, the 1-cocycle relation implies that

$$\omega_1(g, h \cdot y)\, \omega_2(h, z) = \omega_2(h, g \cdot z)\, \omega_1(g, y) \tag{6.2}$$

for all $g \in T_1$, $h \in T_2$ and a.e. $y \in \widehat{S_1}$, $z \in \widehat{S_2}$.

Define the subgroup $G_1 < \Lambda_1$ given by

$$G_1 = \{s \in \Lambda_1 \mid \forall t \in \Lambda_2, tst^{-1} \in \Lambda_1\} .$$

Similarly, define $G_2 < \Lambda_2$. Note that G_1 and G_2 are normal subgroups of Λ_0. Since $\Lambda_0 = \Lambda_1\Lambda_2$ and $\Lambda_1 \cap \Lambda_2 = \{e\}$, we also have that G_1 and G_2 commute. Rewriting (6.2) as

$$\omega_1(g, y)\, \omega_2(h, z) = \omega_2(h, g \cdot z)\, \omega_1(g, h^{-1} \cdot y) ,$$

we find that for all $g \in T_1$, $h \in T_2$ and a.e. $y, y' \in \widehat{S_1}$, $z \in \widehat{S_2}$

$$\omega_2(h, z)^{-1}\, \omega_1(g, y')^{-1}\, \omega_1(g, y)\, \omega_2(h, z) \in \Lambda_1 .$$

Since $L(H_2) = B_2 \rtimes \Lambda_2$, the essential range of ω_2 equals Λ_2. It thus follows that

$$\omega_1(g, y')^{-1}\, \omega_1(g, y) \in G_1$$

for all $g \in T_1$ and a.e. $y, y' \in \widehat{S_1}$. For every $g \in T_1$, we choose $\delta_1(g) \in \Lambda_1$ such that $\omega_1(g, y) = \delta_1(g)$ on a non-negligible set of $y \in \widehat{S_1}$. We conclude that $\omega_1(g, y) = \delta_1(g)\, \mu_1(g, y)$ with $\mu_1(g, y) \in G_1$ a.e. We similarly decompose $\omega_2(h, z) = \delta_2(h)\, \mu_2(h, z)$.

With these decompositions of ω_1 and ω_2 and using that G_1, G_2 are commuting normal subgroups of Λ_0, it follows from (6.2) that for all $g \in T_1$, $h \in T_2$, the commutator $\delta_2(h)^{-1}\delta_1(g)^{-1}\delta_2(h)\delta_1(g)$ belongs to G_1G_2, so that it can be uniquely written as $\eta_1(g, h)\eta_2(g, h)^{-1}$ with $\eta_i(g, h) \in G_i$. It then follows from (6.2) that

$$\begin{aligned} \mu_1(g, h \cdot y) &= \delta_2(h)\, \eta_1(g, h)\, \mu_1(g, y)\, \delta_2(h)^{-1} \quad , \\ \mu_2(h, g \cdot z) &= \delta_1(g)\, \eta_2(g, h)\, \mu_2(h, z)\, \delta_1(g)^{-1} \quad , \end{aligned} \tag{6.3}$$

almost everywhere. Since $S_1 < H_1$ is torsion free, $\widehat{S_1}$ has no finite quotients and thus no proper closed finite index subgroups. It follows that finite index subgroups of T_2 act ergodically on $\widehat{S_1}$. We claim that for every $g \in T_1$, the map $y \mapsto \mu_1(g, y)$ is essentially constant. To prove this claim, fix $g \in T_1$ and denote $\xi : \widehat{S_1} \to G_1 : \xi(y) = \mu_1(g, y)$. For every $h \in T_2$, define the permutation

$$\rho_h : G_1 \to G_1 : \rho_h(s) = \delta_2(h)\, \eta_1(g, h)\, s\, \delta_2(h)^{-1} .$$

So, (6.3) says that $\xi(h \cdot y) = \rho_h(\xi(y))$ for all $h \in T_2$ and a.e. $y \in \widehat{S_1}$. Defining $V_1 \subset G_1$ as the essential range of ξ, it follows that $\{\rho_h\}_{h\in T_2}$ is an action of T_2 on V_1. The push forward via ξ of the Haar measure on $\widehat{S_1}$ is a $\{\rho_h\}_{h\in T_2}$-invariant probability measure on the countable set V_1 and has full support. It follows that all orbits of the action $\{\rho_h\}_{h\in T_2}$ on V_1 are finite. Choosing $s \in V_1$, the set $\xi^{-1}(\{s\}) \subset \widehat{S_1}$ is non-negligible and globally invariant under a finite index subgroup of T_2. It follows that $\xi(y) = s$ for a.e. $y \in \widehat{S_1}$, thus proving the claim.

Similarly, for every $h \in T_2$, the map $z \mapsto \mu_2(h, z)$ is essentially constant. So we have proved that $\omega_1(g, y) = \delta_1(g)$ and $\omega_2(h, z) = \delta_2(h)$ a.e. But then, (6.2) implies that Λ_1 and Λ_2 commute, so that Λ_0 is an abelian group.

Since A_0 is a factor, $B_0^{\Lambda_0} = \mathbb{C}1$ and thus $L(\Lambda_0) \subset A_0$ is maximal abelian. Since $L(\Lambda_0) = L(\Lambda_1) \vee L(\Lambda_2)$ with $L(\Lambda_i) \subset L(H_i)$, the same reasoning as with $B_i \subset L(H_i)$, using (6.1), gives us subgroups $T_i \subset H_i$ such that $L(\Lambda_i) = L(T_i)$ and $T_1 = T_2^{\perp}$, $T_2 = T_1^{\perp}$. Since $L(H_i) = B_i \rtimes \Lambda_i$ with $B_i = L(S_i)$ and $L(\Lambda_i) = L(T_i)$, we get that $H_i = S_i \oplus T_i$.

So far, we have proved that $B_0 = L(S_1) \vee L(S_2)$ and $L(\Lambda_0) = L(T_1) \vee L(T_2)$. In any crossed product $B_0 \rtimes \Lambda_0$ by a faithful action, the only unitaries in $L(\Lambda_0)$ that normalize B_0 are the multiples of the canonical unitaries $\{v_s\}_{s\in\Lambda}$. Therefore, $\mathbb{T}T_1T_2 = \mathbb{T}\Lambda_0$. We have thus proved that the gms decomposition (B_0, Λ_0) is identical to the gms decomposition $(L(S_1) \vee L(S_2), T_1T_2)$.

Since Λ_0, H_1 and H_2 are globally $\{\beta_g\}_{g\in\Gamma_0}$-invariant, it follows that T_i is a globally $\mathrm{SL}(3,\mathbb{Z})$-invariant subgroup of H_i. Thus, $T_i = Q_i^3$ for some subgroup $Q_i < L_i$. Since B_0, H_1 and H_2 are globally Γ_0-invariant, it follows in the same way that $S_i = P_i^3$ for some subgroups $P_i < L_i$. Then, $L_i = P_i \oplus Q_i$ and P_1, P_2, as well as Q_1, Q_2, are each other's orthogonal complement under Ω. So, $L_1 = P_1 \oplus Q_1$ is an admissible direct sum decomposition. $\square$

We now combine Proposition 6.1 with Theorem 5.1 and Proposition 5.14. We fix once and for all $\Gamma = \mathbb{F}_\infty$, $\Gamma_0 = \mathrm{SL}(3,\mathbb{Z})$ and a surjective homomorphism $\beta : \Gamma \twoheadrightarrow \Gamma_0$ so that the automorphism $g \mapsto (g^{-1})^T$ of Γ_0 lifts to an automorphism of Γ. An obvious way to do this is by enumerating $\Gamma_0 = \{g_0, g_1, \ldots\}$ and defining $\beta : \Gamma \to \Gamma_0$ by $\beta(s_i) = g_i$ for $i \geq 0$, where $(s_i)_{i\in\mathbb{N}}$ are free generators of Γ. Note that $\operatorname{Ker}\beta$ is automatically nontrivial.

We also fix countable abelian torsion free groups L_1, L_2 and a dense embedding $L_1 \hookrightarrow \widehat{L_2}$. Put $H_i = L_i^3$ and let Γ act on $H_1 \hookrightarrow \widehat{H_2}$ through β. Then define $A_0 = L^\infty(\widehat{H_2}) \rtimes H_1$ together with the natural action $\beta : \Gamma \curvearrowright (A_0, \tau_0)$. Put $(A, \tau) = (A_0, \tau_0)^\Gamma$ with the action $\Gamma \times \Gamma \curvearrowright (A, \tau)$ given by $(g, h) \cdot \pi_k(a) = \pi_{gkh^{-1}}(\beta_h(a))$ for all $g, h, k \in \Gamma$, $a \in A_0$. Write $M = A \rtimes (\Gamma \times \Gamma)$.

We call an automorphism of L_1 admissible if it extends to a continuous automorphism of $\widehat{L_2}$. We call an isomorphism $\theta : L_1 \to L_2$ admissible if it extends to a continuous isomorphism $\widehat{L_2} \to \widehat{L_1}$.

Theorem 6.2. *Whenever $L_1 = P_1 \oplus Q_1$ is an admissible direct sum decomposition with corresponding $L_2 = P_2 \oplus Q_2$, we define $B(P_1, Q_1) := (L(P_1^3) \vee L(P_2^3))^\Gamma$ and $\Lambda(P_1, Q_1) = (Q_1^3 \oplus Q_2^3)^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$.*

- *Every $B(P_1, Q_1), \Lambda(P_1, Q_1)$ gives a gms decomposition of M.*
- *Every gms decomposition of M is unitarily conjugate with a $B(P_1, Q_1), \Lambda(P_1, Q_1)$ for a unique admissible direct sum decomposition $L_1 = P_1 \oplus Q_1$.*
- *Let $L_1 = P_1 \oplus Q_1$ and $L_1 = P_1' \oplus Q_1'$ be two admissible direct sum decompositions with associated gms decompositions (B, Λ) and (B', Λ').*
  - *(B, Λ) and (B', Λ') are conjugate by an automorphism of M if and only if there exists an admissible automorphism $\theta : L_1 \to L_1$ with $\theta(P_1) = P_1'$, $\theta(Q_1) = Q_1'$, or an admissible isomorphism $\theta : L_1 \to L_2$ with $\theta(P_1) = P_2'$, $\theta(Q_1) = Q_2'$.*
  - *The Cartan subalgebras B and B' are unitarily conjugate if and only if $P_1 = P_1'$.*
  - *The Cartan subalgebras B and B' are conjugate by an automorphism of M if and only if there exists an admissible automorphism $\theta : L_1 \to L_1$ with $\theta(P_1) = P_1'$ or an admissible isomorphism $\theta : L_1 \to L_2$ with $\theta(P_1) = P_2'$.*

Proof. Because of Proposition 6.1, Theorem 5.1 and Proposition 5.14, it only remains to describe all automorphisms $\psi : A_0 \to A_0$ that normalize the action $\beta : \Gamma \curvearrowright A_0$. This action β is defined through the quotient homomorphism $\Gamma \twoheadrightarrow \Gamma_0$. Every automorphism of $\Gamma_0 = \mathrm{SL}(3,\mathbb{Z})$ is, up to an inner automorphism, either the identity or $g \mapsto (g^{-1})^T$. So, we only need to describe all automorphisms $\psi : A_0 \to A_0$ satisfying either $\psi \circ \beta_g = \beta_g \circ \psi$ for all $g \in \Gamma_0$, or $\psi \circ \beta_g = \beta_{(g^{-1})^T} \circ \psi$.

In the first case, reasoning as in the first paragraphs of the proof of Proposition 6.1, we get that $\psi(L(H_i)) = L(H_i)$ for $i = 1, 2$. So, for every $g \in H_1$, $\psi(u_g)$ is a unitary in $L(H_1)$ that normalizes $L(H_2)$. This forces $\psi(u_g) \in \mathbb{T}H_1$ and we conclude that $\psi(\mathbb{T}H_1) = \mathbb{T}H_1$. Similarly, $\psi(\mathbb{T}H_2) = \mathbb{T}H_2$. In the second case, we obtain in the same way that $\psi(\mathbb{T}H_1) = \mathbb{T}H_2$ and

$\psi(\mathbb{T}H_2) = \mathbb{T}H_1$. The further analysis is analogous in both cases and we only give the details of the first case.

We find automorphisms $\theta_i : H_i \to H_i$ such that $\psi(u_g) \in \mathbb{T}u_{\theta_i(g)}$ for all $i = 1, 2$ and $g \in H_i$. Since θ_i commutes with the action of $\mathrm{SL}(3, \mathbb{Z})$ on H_i, we get that $\theta_1 = \theta^3$, where $\theta : L_1 \to L_1$ is an admissible automorphism. It follows that ψ maps the gms decomposition associated with $L_1 = P_1 \oplus Q_1$ to the gms decomposition associated with $L_1 = \theta(P_1) \oplus \theta(Q_1)$. This concludes the proof of the theorem. □

The following concrete examples provide a proof for Theorem A.

Theorem 6.3. *For all $n \geq 1$, consider the following two embeddings $\pi_i : \mathbb{Z}^n \hookrightarrow \mathbb{T}^{2n}$.*

- $\pi_1(k) = (\alpha_1^{k_1}, \alpha_2^{k_1}, \ldots, \alpha_{2n-1}^{k_n}, \alpha_{2n}^{k_n})$ *for rationally independent irrational angles $\alpha_j \in \mathbb{T}$.*
- $\pi_2(k) = (\alpha^{k_1}, \beta^{k_1}, \ldots, \alpha^{k_n}, \beta^{k_n})$ *for rationally independent irrational angles $\alpha, \beta \in \mathbb{T}$.*

Applying Theorem 6.2 to the embeddings π_1 and π_2, we respectively obtain

- *a II$_1$ factor M that has exactly 2^n gms decompositions up to unitary conjugacy, and with the associated 2^n Cartan subalgebras non conjugate by an automorphism of M ;*
- *a II$_1$ factor M that has exactly $n+1$ gms decompositions up to conjugacy by an automorphism of M, and with the associated $n+1$ Cartan subalgebras non conjugate by an automorphism of M.*

Proof. Whenever $\mathcal{F} \subset \{1, \ldots, n\}$, we have the direct sum decomposition $\mathbb{Z}^n = P(\mathcal{F}) \oplus P(\mathcal{F}^c)$ where $P(\mathcal{F}) = \{x \in \mathbb{Z}^n \mid \forall i \notin \mathcal{F}, x_i = 0\}$.

In the case of π_1, these are exactly all the admissible direct sum decompositions of $\mathbb{Z}^n$. Also, the only admissible automorphisms of $\mathbb{Z}^n$ are the ones of the form $(x_1, \ldots, x_n) \mapsto (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$ with $\varepsilon_i = \pm 1$. Since $\mathbb{Z}^n \ncong \mathbb{Z}^{2n}$, there are no isomorphisms "exchanging L_1 and L_2".

In the case of π_2, all direct sum decompositions and all automorphisms of $\mathbb{Z}^n$ are admissible. For every direct sum decomposition $\mathbb{Z}^n = P_1 \oplus Q_1$, there exists a unique $k \in \{0, \ldots, n\}$ and an automorphism $\theta \in \mathrm{GL}(n, \mathbb{Z})$ such that $\theta(P_1) = P(\{1, \ldots, k\})$ and $\theta(Q_1) = P(\{k + 1, \ldots, n\})$. Again, there are no isomorphisms exchanging L_1 and L_2. So, the $n + 1$ direct sum decompositions $\mathbb{Z}^n = P(\{1, \ldots, k\}) \oplus P(\{k + 1, \ldots, n\})$, $0 \leq k \leq n$, exactly give the possible gms decompositions of M up to conjugacy by an automorphism of M. When $k \neq k'$, there is no isomorphism $\theta \in \mathrm{GL}(n, \mathbb{Z})$ with $\theta(P(\{1, \ldots, k\})) = P(\{1, \ldots, k'\})$. Therefore, the $n + 1$ associated Cartan subalgebras are nonconjugate by an automorphism either. □

Remark 6.4. When L_1, L_2 are torsion free abelian groups and $L_1 \hookrightarrow \widehat{L_2}$ is a dense embedding, then the set of admissible homomorphisms $L_1 \to L_1$ is a ring $\mathcal{R}$ that is torsion free as an additive group. The admissible direct sum decompositions of L_1 are in bijective correspondence with the idempotents of $\mathcal{R}$. As a torsion free ring, $\mathcal{R}$ either has infinitely many idempotents, or finitely many that are all central, in which case their number is a power of 2. So, the number of gms decompositions (up to unitary conjugacy) of the II$_1$ factors produced by Theorem 6.2 is always either infinite or a power of 2.

Remark 6.5. Still in the context of Theorem 6.2, we call a subgroup $P_1 < L_1$ admissible if $L_1 \cap \overline{P_1} = P_1$, where $\overline{P_1}$ denotes the closure of P_1 inside $\widehat{L_2}$. Note that $P_1 < L_1$ is admissible if and only if there exists a subgroup $P_2 < L_2$ such that $P_2 = P_1^\perp$ and $P_1 = P_2^\perp$. Whenever $P_1 < L_1$ is an admissible subgroup, we define $B(P_1) := (L(P_1^3) \vee L(P_2^3))^\Gamma$. It is easy to check that all $B(P_1)$ are Cartan subalgebras of M and that $B(P_1)$ is unitarily conjugate with $B(P_1')$ if and only if $P_1 = P_1'$.

It is highly plausible that these $B(P_1)$ describe all Cartan subalgebras of M up to unitary conjugacy. We could however not prove this because all our techniques make use of the dual coaction associated with a gms decomposition of M.

Also note that an admissible subgroup $P_1 < L_1$ cannot necessarily be complemented into an admissible direct sum decomposition $L_1 = P_1 \oplus Q_1$. In such a case, $B(P_1)$ is a Cartan subalgebra of M that is not of group measure space type. However, M can be written as a *cocycle* crossed product of $B(P_1)$ by an action of $\Lambda(P_1) = \big((L_1/P_1)^3 \oplus (L_2/P_2)^3\big)^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, but the cocycle is nontrivial.

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